Integral Transforms and Their Applications
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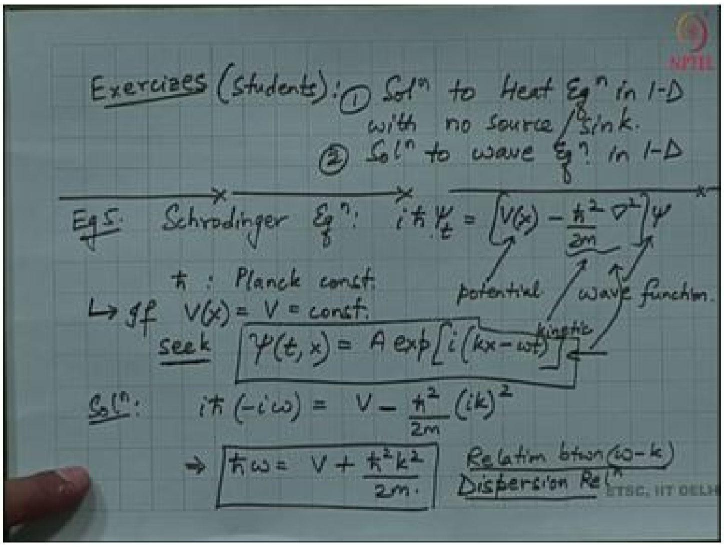
Lecture – 3
Applications of Fourier Transforms Part - 03

So, next I am going to talk about another case another example. So, in particular I have this famous equation given by Schrodinger. So, this is another wave equation, which tells us notice that this is now a complex wave because of the presence of this i.



Schrodinger Equation:

$$i\hbar\psi_t = \left[V(x) - \frac{\hbar^2}{2m}\nabla^2\right]\psi$$

where, ψ=Wave function, $V(x)$ = Potential and $\frac{\hbar^2}{2m}\nabla^2$=Kinetic Energy

So, typically I have this Schrodinger equation which measures the total energy of a particle given by the sum of this potential energy. so, just to give a brief overview of this equation. So, suppose then I have this constant h with a cross it is also known as the Planck's constant and this has a known value. So, if I have; if I choose my potential v to be a constant right. So, if my potential v is a constant, then I can find the solution of this example of this wave function of the type of plane waves. So, what do I mean by that? If I take my v to be constant I can show that my solution to this Schrodinger equation is of this form.

Now, let us look at. So, let us look at the solution to this equation with v constant. So, I am going to assume that v is constant. So, this is going to be assumed and I am going to solve this equation, ok. So, if I were to apply my if I were to choose solutions of this form. So, seek solutions of this form and for that type of solutions I would like to know what is this A, this amplitude A.

So, if I plug this; if I plug this solution to this equation; this Schrodinger equation with v constant I get the following expression:

$$\psi(t, x) = A\exp[i(kx - \omega t)]$$

Solution:

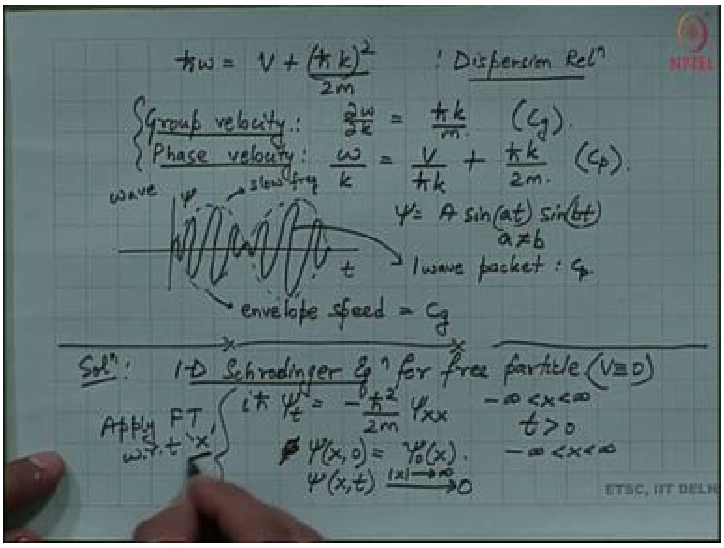
$$i\hbar(-i\omega) = V - \frac{\hbar^2}{2m}(ik)^2$$

So, ∇^2 square it is a problem in one dimension. So, ∇ each derivative of x is going to pull out one i k. So, I get i k square. So, if I were to simplify; I get this is equal to,

$$\hbar\omega = V + \frac{\hbar^2 k^2}{2m}$$

Notice that this is the relation between the relation between omega and k. So, if I were to choose my solution of this form, ω and k must satisfy this following relation if this function is to be a solution to this equation. Now in particular this relation or relation to any PDEs in general are also called as the dispersion relation. So, when I talk about the PDEs and I transform the PDEs into this Fourier plane, then the corresponding PDE that I or the corresponding equation that I get in the transformed Fourier plane, Fourier or Fourier Laplace plane is the so called dispersion relation.

So, this is identically equal to the PDE in the physical plane or the x-t plane. So, this is quite common; it is quite common by scientist throughout the world to figure out the dispersion relation and compare it with the solution to the PDE and how accurate is the solution.



Now, just to give you a bit of bit more highlight. We see that; we see that for the dispersion relation let me write down:

$$\hbar w = v + \frac{(\hbar k)^2}{2m} \quad \therefore \text{ Dispersim Relation}$$

Now, there are some important quantities that can be found out of this dispersion relation. One of them being the so called group velocity right and the other one that is commonly used is the so called phase velocity. So, what is group velocity? Well, mathematically group velocity is given by the expression:

$$\frac{\partial w}{\partial k} = \frac{\hbar k}{m}$$

So, I need to divide throughout by h cut here and differentiate omega with respect to k. So, in this case if I were to find the group velocity, I get that this is equal to:

$$\frac{w}{k} = \frac{v}{\hbar k} + \frac{\hbar k}{2m}$$

Notice that the group velocity and the phase velocity are two separate quantities. Now, let me just give you a little bit of highlight as to what are these quantities. So, suppose I am talking about a wave. Let us say I have a wave which is moving with time. So, I have a wave, a wave. Let us say

$$\psi = A\sin(at)\sin(bt) \quad \textbf{where,} a \neq b$$

So, notice that this wave has two frequencies. Well, one will be say let say a is not equal to b. So, in this scenario there will be two frequencies with which this wave is propagating with time or changing with time. One could be where a is bigger, in that case we have the slow wave and when b being smaller we could have the fast wave.

So, if I were to plot this expression over this on this graph(shown in the slide) I am going to get the following curve. So, let us say this is my envelope. So, this is my slow wave this is my slow the slow frequency evolution and within that slow frequency I will get the fast evolution of the wave; so, the speed of this. So, this is the envelope. So, the envelope speed, the envelope speed is also known as the group velocity c_g.

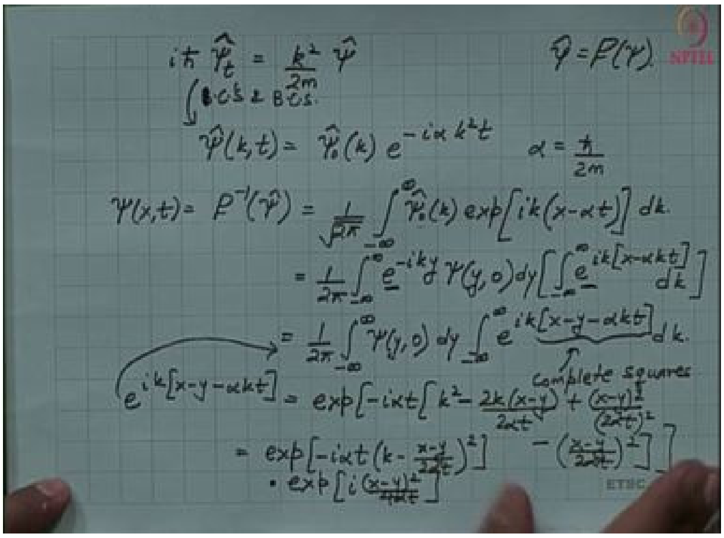
So, I term this as c_g, the group velocity and the velocity of this 1 wave packet is also called as the phase velocity. So, I denote this by c p. So, the velocity of 1 wave packet or 1 particle within that wave is the phase velocity and the velocity of the entire envelope of these waves is the group velocity, ok. Moving on to the solution, moving on to the solution I have the following. So, I am solving 1-D Schrodinger equation just to recap. I am solving 1-D Schrodinger equation for free particles, 1-D Schrodinger equation for free particles that is v is identically equal to 0. So, what I have is that, I have the following equation.

Solution:

$$i\hbar\psi_t^N = -\frac{\hbar^2}{2m}\psi_{xx} \qquad \begin{matrix} -\infty & < x < \infty \\ t > 0 & \end{matrix}$$

$$\psi(x,0) = \psi_0(x) \cdot \qquad -\infty < x < \infty$$
$$\psi(x,t) \overset{(x)\to\infty}{\longrightarrow} 0$$

So, to solve this, if I were to apply the Fourier transform, so apply Fourier transform with respect to the variable x. So, I apply Fourier transform with respect to the variable x.



So, I get the following expression.

$$i\hbar\hat{\psi}_t = \frac{k^2}{2m}\hat{\psi} \quad \text{where,} \quad \widehat{\psi} = F.T(\psi)$$

Now, after applying the boundary conditions,I get the solutions $\widehat{\psi}$ of the following form:

$$\hat{\psi}(k,t) = \hat{\psi}_0(k)e^{-i\alpha k^2 t} \quad \alpha = \frac{\hbar}{2m}$$

Notice that as my t goes to infinity, well, at t equal to 0, the solution reduces to the solution given by the initial condition and as t goes to infinity; we see that the solution decays. So, this satisfies all the initial and the boundary conditions. So, then my solution:

$$\psi(x,t) = (F.T)^{-1}(\widehat{\psi}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{\psi_0}(k)\exp[ik(x-\alpha t)]dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iky}\psi(y,0)dy\left[\int_{-\infty}^{\infty} e^{ik[x-\alpha kt]}dk\right]$$

Now, I have two integral, one with respect to y the other with respect to k. So, let us now combine these exponential factors and rewrite our expressions. So, this becomes:

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi(y,0)dy\int_{-\infty}^{\infty} e^{ik[x-y-\alpha kt]}dk$$

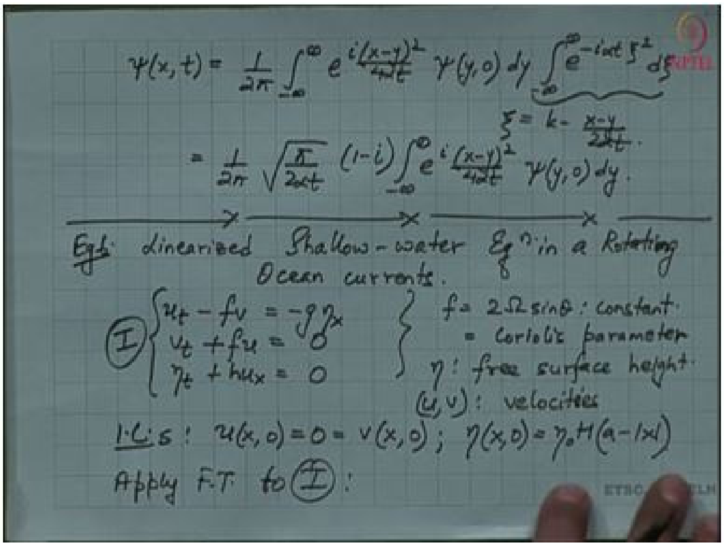
So, I am going to evaluate this second integral by completing the squares. So, I am going to complete squares here.

$$e^{ik[x-y-\alpha kt]} = \exp\left[-ikt\left[k^2 - \frac{2k(x-y)}{2\alpha t} + \frac{(x-y)^2}{(2\alpha t)^2} - \frac{(x-y)^2}{(2\alpha t)^2}\right]\right].$$

So, after completing the square I am going to see that,

$$= \exp\left[-i\alpha t\left(k - \frac{x-y}{2\alpha t}\right)^2\right].\exp\left[i\frac{(x-y)^2}{4\alpha t}\right]^2$$

So, this expression is going to be substituted in this integral and after substituting these values.



$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{i(x-y)^2}{4\alpha t}}\psi\left(y,0\right)dy \int_{-\infty}^{\infty} e^{-i\alpha t\xi^2}d\xi \quad \text{where,} \quad \xi = k - \frac{x-y}{2\alpha t}$$

And finally, after evaluating this integral, I am going to get the following expression that this is:

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{2\alpha t}}(1-i)\int_{-\infty}^{\infty}\frac{e^{i(x-y)^2}}{4\alpha t}\psi(y,0)dy$$

that is the solution to this problem, which is left in the form of one integral depending on whatever this function ψ at 0 will be ok.

Then, I am going to talk about another short example that is on the case scenario of linearized shallow water, linearized shallow water equation. So, in specially found in rotating ocean currents, rotating ocean currents. Now, it has been found that if we were to track the velocity of the water waves, the velocity of the water waves on the surface of the ocean and treating the ocean; the ocean floor as a two dimensional surface with the height separately treated as a variable. Then, I am able to write the equation for these wave motion in the linearized format as follows.

So, I have the following equations:

$$(\text{I}) \left\{\begin{array}{l} u_t - f_v = -g\eta_x \\ v_t + f_u = 0 \\ \eta_t + hu_x = 0 \end{array}\right\}$$

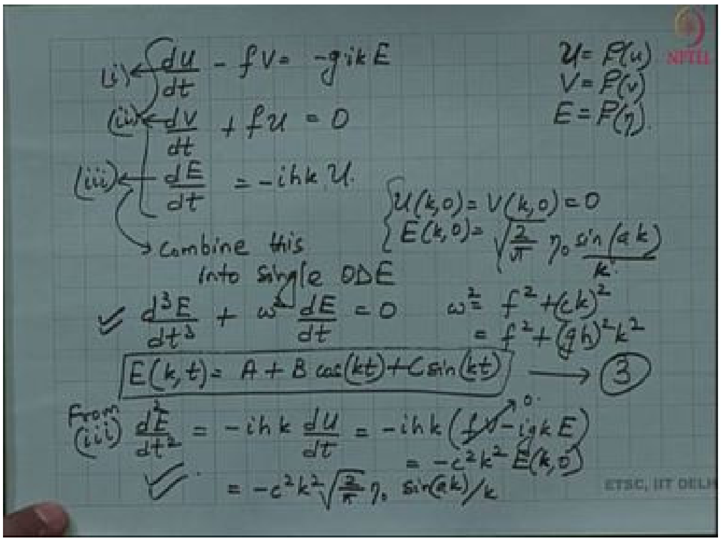
$$\text{where,} \quad f = 2\Omega\sin\theta : \text{ constant } = \text{coriolis parameter}$$

$$\eta : \text{free surface height.} \quad (u,v) : \textit{velocities}$$

So, then, I need also some initial conditions, some initial conditions which are given by

$$u(x,0) = 0 = v(x,0); \quad \eta(x,0) = \eta_0 H(a - |x|)$$

Now, if we were; let us say that the set of equations is 1. So, I am going to apply Fourier transform to 1. So, applying Fourier transform to 1, I get the following new set of equations.



Apply Fourier Transform to (1),I get:

$$\begin{array}{lcl} \text{(i) } \frac{du}{dt} - fv = -gikE & & u = \text{F.T}(u) \\ \text{(ii) } \frac{dV}{dt} + fu = 0 & \text{where,} & V = \text{F.T}(v) \\ \text{(ii) } \frac{dE}{dt} = -ihkU. & & E = \text{F.T}(\eta) \end{array}$$

So, if I were to combine, if I were to combine these three ODEs. Notice that, this is now an ODE and the initial conditions of the ODEs are:

$$u(k,0) = V(k,0) = 0$$
$$E(k,0) = \sqrt{\frac{2}{\pi}}\eta_0 \frac{\sin(ak)}{k}$$

we take the Fourier transform of the initial conditions for E and that is what we get; so, combining this into a single ODE. So, notice that this set of three ODEs I can combine. Specifically, I can take the derivative of E, which will be as a function of the derivative of u. So, we need to take further another derivative of E to give me dV/dt and so that is how we eliminate and come up with a single equation for E, which is given by the following;

$$\frac{d^3E}{dt^3} + \omega^2 \frac{dE}{dt} = 0$$

$$\text{where,} \quad \begin{aligned} \omega^2 &= f^2 + (ck)^2 \\ &= f^2 + (gh)^2 k^2 \end{aligned}$$

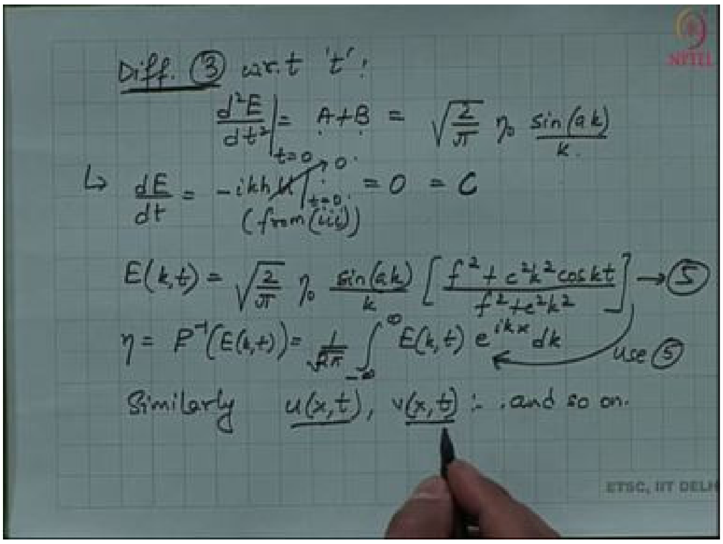
$$E(k,t) = A + B\cos(kt) + C\sin(kt) \rightarrow (3)$$

You can check. We can check that this is indeed the solution to this ODE, which satisfies. Now, we must make sure that this solution satisfies these initial conditions. So, when; so, in particular if I take the second derivative of E with respect to t, I get that this is equal to:

$$\frac{d^2E}{dt^2} = -ihk\frac{du}{dt} = -ihk\,(fv - igkE)$$

$$= -c^2k^2E(k, 0)$$

$$= -c^2k^2\sqrt{\frac{2}{\pi}}\eta_0 \sin(\alpha k)$$



So, if I were to differentiate 3 with respect to t, I get that:

$$\frac{d^2E}{dt^2} = A + B = \sqrt{\frac{2}{\pi}}\eta_0 \frac{\sin(ak)}{k} \quad \text{for,t} = 0$$

So, that is one relation between A and B. Then, what I have is another relation is that

$$\frac{dE}{dt} = -ikhu = 0 = c \quad \text{from equation (3)}$$

So, this is evaluated at t equal to 0. I get that this is equal to 0 because u is 0 and I get that again finding dE/dt from my expression 3; I get that this is equal to C. So, after finding A, B, C in this fashion, I get that my E my expression:

$$E(k, t) = \sqrt{\frac{2}{\pi}}\eta_0 \frac{\sin{(\alpha k)}}{k} \left[\frac{f^2 + c^2k^2 \cos kt}{f^2 + c^2k^2}\right] \to (5)$$

So then η which will be the Fourier inverse of this expression for E will be given by the inverse transform. So, we take the inverse transform:

$$\eta = (F.T)^{-1}(E(k, t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(k, t)e^{ikx}dk$$

I use this expression 5, I get my answer and the answer will be only in terms of; so, use 5. So, use this expression 5 to come up to this answer for η. Now, similarly I can find, similarly I can find expression for u for v and so on; so and so on.

$$\text{Similary } u(x,t), v(x,t): \text{ and so on.}$$

Well, these are the only two variables which remain.

So, you can see that this particular example shows us how to calculate the solution to this rotating ocean currents in which we have linearized the coefficients of the equation using our Fourier transform. Of course, if we know our initial condition we can exactly solve all these integral to get an exact analytical expressions for our answer. So, in the next lecture I am going to talk about again another application of Fourier transforms and further from moving from there onwards I am going to talk about Laplace transform. I will introduce Laplace transform and then I will talk about some properties of Laplace transform and later on we will look at some interesting applications of Laplace transform. Thank you very much.