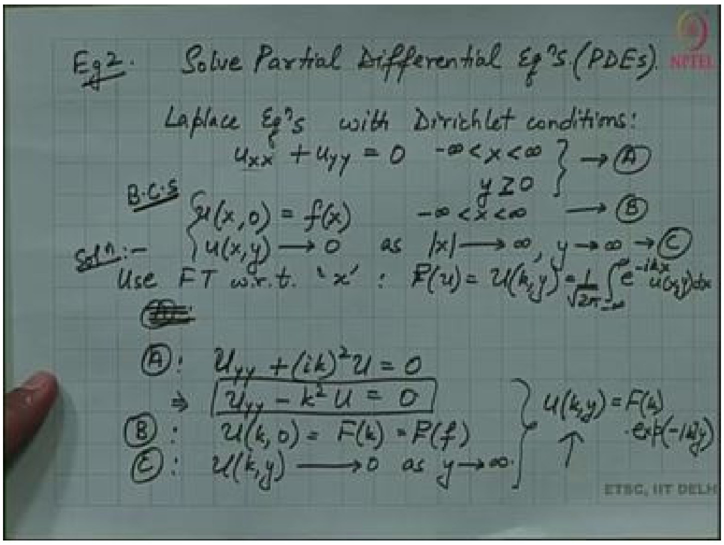
Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture – 3
Applications of Fourier Transforms Part - 02

So, here is one solve this partial differential equation; solve this partial differential equation. So, what I have is. So, I from now on I am going to denote my partial differential equation as PDE. So, PDEs denote partial differential equation. So, I have the so called Laplace equation. So, what is Laplace equation; a Laplace equation with Dirichlet condition.



So, I have the equation of this form:

$$u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty \qquad (A)$$

So, this is an equation to be solved over half of the plane plane from y=0 to infinity and $-\infty < x < \infty$.Further since this is a boundary value problem. What do I mean that there is no time derivative here.

So, I need two boundary conditions. So, my boundary conditions I am going to talk about boundary conditions and denote it as B C S (as shown in slide). My boundary conditions are as follows: B.C.S:

$$u(x,0) = f(x) \quad -\infty < x < \infty \qquad (B)$$

$$u(x,y) \to 0 \quad \text{as} \quad |x| \to \infty, y \to \infty \qquad (C)$$

So, what this solution says is that on the surface on y equal to 0 you have certain value of the solution and the solution decays as your variables go to infinity, your independent variables go to infinity. So, in the far field or faraway from y equal to 0 the solution decays to 0. So, I am going to; so let us start to figure out the solution to this Laplace equation I am going to use the Fourier transform with respect to the variable x right. So, I have two variable x independent variables x and y this question is for which variable should I use the Fourier transform right.

So, you can see that x is varying from negative infinity to infinity while y we only have non-negative values. So, it is natural quite natural to use the Fourier transform for the variable f with respect to the variable x. So, then let us say this is my equation A this is my condition B and this is my condition C. So, my A reduces to the following. So, well let me just say that the Fourier transform of u is denoted by:

$$F(u) = u(k,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} u(x,y)dx$$

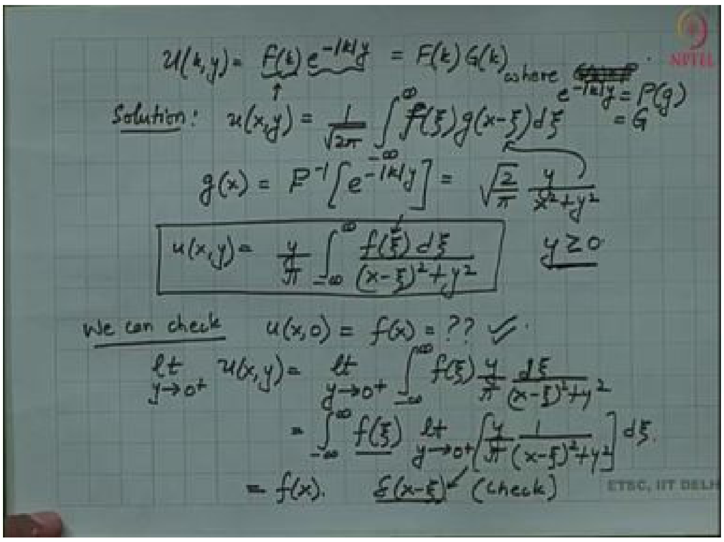
So, then my A is reduced to the following. So, if I were to apply the Fourier transform and use the derivative for the derivative results for Fourier transform I get that:

$$\text{(A): } u_{yy} + (ik)^2 u = 0$$
$$\Rightarrow u_{yy} - k^2 u = 0$$

My B application of Fourier transform to be gives:

$$\left.\begin{array}{l}\text{(B): u(k,0)=F(k) =F(f)} \\ \text{(C): } u(k,y) \to 0 \text{ as } y \to \infty\end{array}\right\} u(k,y) = F(k).exp(-|k|y)$$

but the Fourier transform of the small f and my application of Fourier transform to C tells me that u(k,y) goes to 0 as y goes to infinity. So, here I have an equation to solve this is an ODE an Ordinary Differential Equation with the following second order ordinary differential equation with the above two conditions. So, this can be easily solved and the solution that I get is $u(k,y) = F(k).exp(-|k|y)$ Notice that as y goes to infinity the solution goes to 0 and at y equal to 0 the solution u(k,y) is f(k). So, it satisfies both the boundary conditions and it is you can it can be checked it is the solution to this ODE. So, then I have to use the inverse transform to this function.



So, I have:

$$u(k,y) = F(k)e^{-|k|y} = F(k)G(k)$$

So, what I have is well I should write the other way where,

$$e^{-|k|y} = F(g) = G$$

So, that that is what I mean by this product. So, then my solution u, since this is a product of two Fourier transform my solution will be the convolution of the corresponding Fourier transform inverse of these two functions.:

$$u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) g(x-\xi) d\xi$$

So, this is the convolution of the corresponding to functions for which we have the Fourier transforms F and G. Now, so I have the following that g is nothing, but the Fourier transform inverse denoted as:

$$g(x) = F^{-1}\left[e^{-|k|y}\right] = \sqrt{\frac{2}{\pi}} \frac{y}{x^2+y^2}$$

So, I am going to use this expression here to come to the solution.So, u(x,y) is y after simplifying I get:

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(x-\xi)^2+y^2}, y \geq 0$$

Where I know that y is non negative. So, this is where the solution can be left because we do not know what is this function y f. Now, we could check we could. So, we can check what happens to u(x,0).whether u(x,0) indeed gives us f(x) or not right.So, check whether this is true or not. So, to do that we take the limit y tending to 0 from the positive end because y is non negative.

$$\lim_{y \to 0^+} u(x,y) = \lim_{y \to 0^+} \int_{-\infty}^{\infty} f(\xi) \frac{y}{\pi} \frac{d\xi}{(x-\xi)^2+y^2}$$

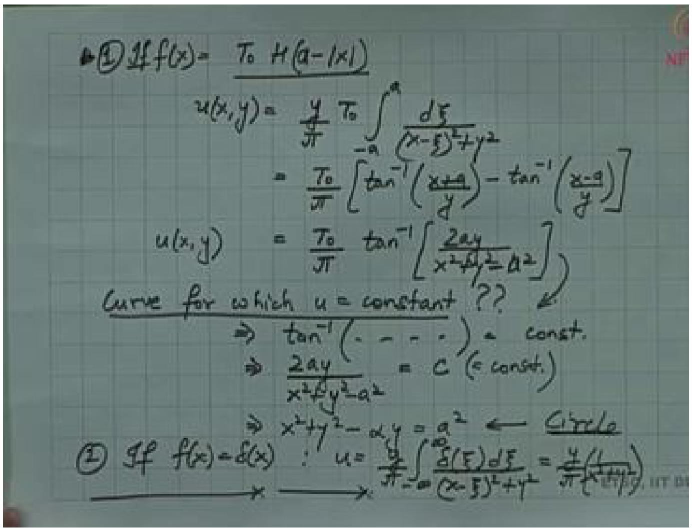
So, let me just rewrite this integral. So, this becomes

$$\lim_{y \to 0^+} u(x,y) = \int_{-\infty}^{\infty} f(\xi) \lim_{y \to 0^+} [\frac{y}{\pi} \frac{d\xi}{(x-\xi)^2+y^2}] d\xi$$

Now, it can be seen that this is also the delta function delta of x minus zeta. So, you need to check that in the limit y going to 0 plus this function reduces to $\delta(x-\xi)$.So, finally, my integral reduces to

$$= f(x)\delta(x-\xi)$$

and by the property of delta function the value from negative infinity to infinity is nonzero only when $\xi = x$. So, all I get is that this integral is f(x). So, the check is done. So, now let us look at some particular cases of this function. So, let us choose some particular cases of f and see what is the solution.



let us say I choose my f(x). So, if my f(x) is given by this heaviside function. So, this is a constant heaviside function is $a - |x|$ right. So, in that case my solution u(x,y) will be:

$$u(x, y) = \frac{y}{\pi} T_0 \int_{-a}^{a} \frac{d\xi}{(x-\xi)^2 + y^2}$$

So, now this integral being finite this can be evaluated. Particularly this integral is nothing, but:

$$u(x, y) = \frac{T_0}{\pi}\left[\tan^{-1}\left(\frac{x+a}{y}\right) - \tan^{-1}\left(\frac{x-a}{y}\right)\right]$$

or we can use the formula for $\tan^{-1}$. So, $\tan^{-1} a - \tan^{-1} b$ gives me the following simplified expression.

$$u(x, y) = \frac{T_0}{\pi}\tan^{-1}\left[\frac{2ay}{x^2 + y^2 - a^2}\right]$$

. So, as far as the answer is concerned this is the answer that I get for u(x,y) for this particular case of my initial my boundary condition. Now, suppose I were to see the curves for which u is constant. So, which are those curves. So, which means that in this case I have that tan inverse this expression; so I am using this expression here.

$$\text{Curve for which } u = \text{constant??}$$
$$\Rightarrow \tan^{-1}(\ldots\ldots) = \text{ const.}$$

So, tan inverse something is a constant it is a constant, only if I have that

$$\Rightarrow \frac{2ay}{x^2 + y^2 - a^2} = c (= \text{const.})$$

So, this is equal to a constant right. So, what I get is that this is also equal to:

$$\Rightarrow x^2 + y^2 - \alpha y = a^2 \leftarrow \text{ circle}$$

we can check after completing the squares that this is a circle this is a circle right. So, the curves for which u is a constant a is a circle. So, these are the level curves for the problem.

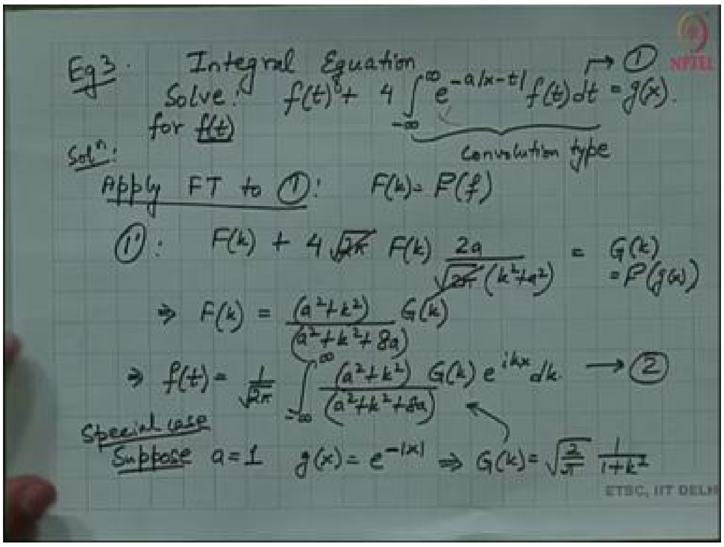
Now, another case could be that if I choose my f(x) to be the delta function then I can see that u again I replace which:

$$(2) \text{ If } f(x) = \delta(x):$$

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\delta(\xi)d\xi}{(x-\xi)^2+y^2}$$

$$= \frac{y}{\pi}\left(\frac{1}{x^2+y^2}\right)$$

So, these are some special cases of the problem any way. So, let us continue and look at some more examples.



So, then I have another example. So, then I have another example of the integral type. So, I have an integral equation an integral equation. So, I need to solve this integral equation which is of the form:

$$f(t) + 4\int_{-\infty}^{\infty} e^{-a|x-t|} f(t)dt = g(x)$$

So, notice that solve I need to solve for f(t). So, notice that f(t) appears here and also f(t) appears inside the integral right. So, I need to solve for this integral equations; also notice that this integral is of convolution type right. So, what I have is that if I were to choose this as my f and this as my g you can notice that this is exactly of the convolution type that we have described earlier. So, which means that if I were to let us say that this is my equation 1. So, I were to apply solution if I were to apply Fourier transform to 1 right. So, I am going to get for the first case I am going to get F(k) where F(k) is the Fourier transform of small f.

$$\text{Apply FT to 1}: \quad F(k) = F.T(f)$$

$$F(k) + 4\sqrt{2\pi}F(k)\frac{2a}{\sqrt{2\pi}\,(k^2+a^2)} = G(k) = F.T(g(x))$$

Fourier transform of this function which has already been done in the last few lectures and take the product of the two Fourier transforms. So, let me call this as 1 prime. So, 1 prime becomes:

$$\Rightarrow F(k) = \frac{(a^2+k^2)}{(a^2+k^2+8a)} G(k)$$

So, if we were to know what is the right hand side; we can immediately take the inverse transform and find the solution F. So, in particular my f(t) is

$$\Rightarrow f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{(a^2+k^2)}{(a^2+k^2+8a)} G(k) e^{ikx} dk \rightarrow (2)$$

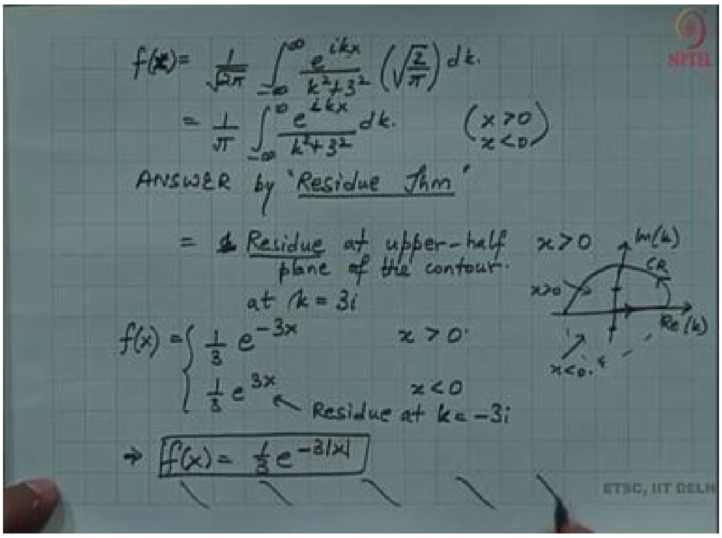
So, let us now look at some specific cases let us say suppose I choose:

$$a = 1, \quad g(x) = e^{-|x|}$$

So, in that case my G(k) the Fourier transform of g(x) is the following expression:

$$G(k) = \sqrt{\frac{2}{\pi}} \frac{1}{1+k^2}$$

So, substituting; so this is a special case. So, substituting all these values here let us say this is my 2; I get the following.



$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+3^2} (\sqrt{\frac{2}{\pi}}) dk$$

So, after I do all the simplification I get that this is equal to

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ikx}}{k^2+3^2} dk$$

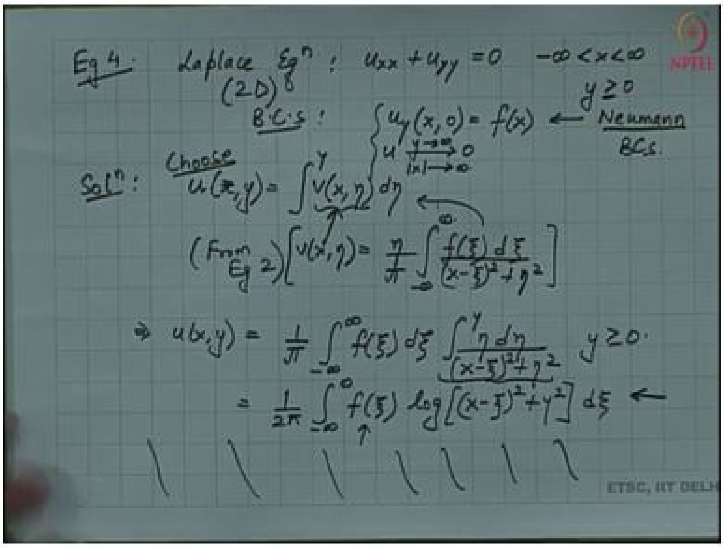
So, notice that again I can find I can find the answer I can find the answer by using residue theorem again going back to our complex variables basics the answer to this expression is that we have to again consider a semicircular arc and see that over the circular part the integral vanishes in the limit in the limit of these two values of this integral and all I am left with is the real integral and if we were to find the answer by the residue theorem it is given by the following. So, this will be. So if I have let us say there are two cases now right. So, I can have either x positive or x negative right it is an either or situation. So, if my x is positive. So, in that case I have to find residue at the upper half plane the upper half plane of the contour and in that case my residue is that. So, this will be at this following (refer the slide) So, this will be at this following value k=3i; notice that I can factor this into (k+3i)(k-3i) right. So, the factor that gives me a positive root will be the one that contributes to the residue and I get calculating the residue I get that f(x). So, this will be f(x). So, f(x) because it is evaluated at x. So, f(x) is after doing all the simplification it is:

$$f(x) = \begin{cases} \frac{1}{3}e^{-3x} & x > 0 \\ \frac{1}{3}e^{3x} & x < 0 \end{cases}$$

again this will be by evaluating the residue at the other root. K-3i because the contour will lie I am talking about the contour cR(Refer image in the slide). So, if I were to combine these two results I am going to get the answer to be:

$$f(x) = \frac{1}{3}e^{-3|x|}$$

So, that is my answer to this problem.



So, continuing then I have let me look at another case, where I am looking at again the Laplace equation the Laplace equation which is given by this following equation:

$$u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty, \quad y \geq 0$$

So, this is Laplace equation in two dimensions and again I choose my x to vary from $-\infty < x < \infty$ and y is from 0 to ∞. So, y is non-negative and x is an can take any value and this time I choose my boundary conditions to be of the following form:

$$\begin{cases} u_y(x,0) = f(x) \leftrightarrow \text{ Neumenn BCS} \\ u \xrightarrow[|x|\to 0]{y\to 0} 0 \end{cases}$$

So, this is in particular we call this boundary condition as Neumann boundary conditions. So, when I have a boundary condition in which the derivative is specified that is the Neumann boundary condition in one of the previous examples we had the value specified at the boundary. So, that is the Dirichlet boundary condition. Now, the only difference between this example and say example 2 where we were using the Dirichlet condition is due to this boundary condition Neumann. So, which means that if I were to choose my u to be of this form. So, choose u (x,y) to be of the form:

$$u(x,y) = \int_0^y v(x,\eta)d\eta$$

So, we can see that if we were to take the derivative the derivative will satisfy the Neumann condition implying that this particular function v will satisfy the Dirichlet condition. So, if we were to choose this as my solution and if v satisfies the Dirichlet condition then u satisfies the corresponding Neumann boundary condition because u is the integral of v. So, which means we have already found the solution to v in particular:

$$v(x,y) = \frac{\eta}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)d\xi}{(x-\xi)^2+\eta^2}d\eta$$

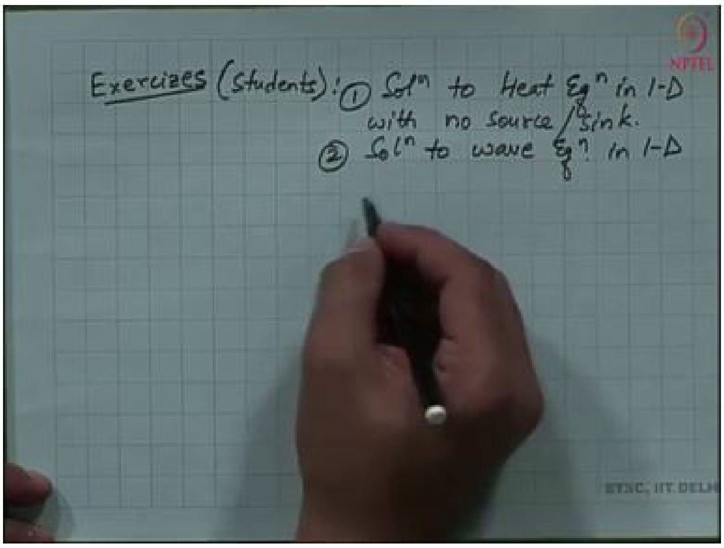
So, notice my variable y is replaced by this variable η in the previous example. So, I am going to substitute in this integral and I am left with the following integration:

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_0^y \frac{\eta d\eta}{(x-\xi)^2+\eta^2}, \quad y \geq 0$$

So, I have just rearranged some of the integration integrals and this is what I get. So, I have that y is non negative. Now, this integral can be readily evaluated we can see that this integral is with respect to η and I am going to get my answer as:

$$= \frac{1}{2\pi}\int_{-\infty}^{0} f(\xi)\log\left[(x-\xi)^2+y^2\right]d\xi$$

. So, from so further evaluation of this integral will depend upon what value of f is provided. So, this answer can be left at this point because we do not have further information about y. So, before I move on to another example I would like to mention.



So, possible exercises for the students that can be done along the same ideas that has been highlighted in this in these four examples. So, in particular the students can look at solutions to heat equation; please see online what do I mean by heat equation. So, heat equation in let us say 1 dimension in 1-D with no source or sink. So, this is a very simple second order partial differential equation and the same idea of Fourier transform can be used to find the solution to the heat equation or another exercise could be the solution to the wave equation in 1-D. So, this can all be found out by the method just described and by the examples just described.