Integral Transform and Their Applications

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Lecture - 71

Discrete Haar, Shanon and Debauchies Wavelet- Part 02

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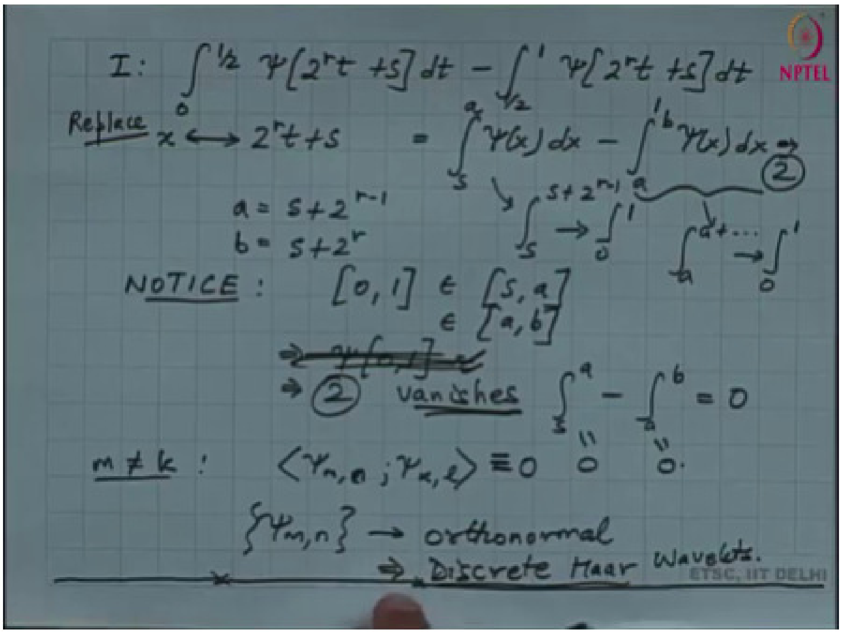

So now, let us look at another example.

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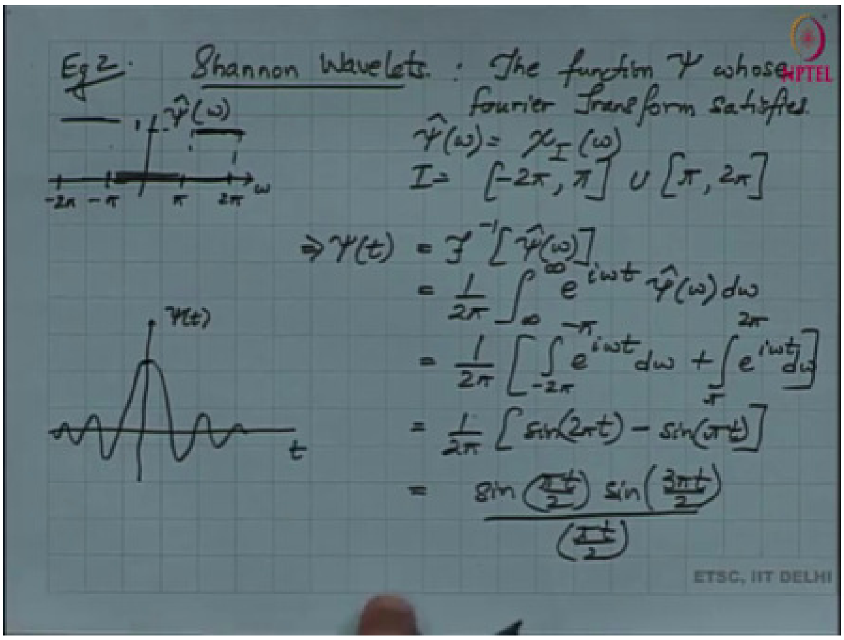

So, let me show you another example of Shannon wavelets. Let me just describe what are these wavelets, very widely used in signal processing. So, Shannon wavelets are wavelets the function ψ whose Fourier transform satisfies the following condition. So, I have

$$\hat{\psi}(w) = \chi_I(w); \quad I = \big[-2\pi, \pi\big] \cup \big[\pi, 2\pi\big]$$

So, if I were to plot, my Fourier transform of the Shannon wavelet, I see that the Fourier transform of the Shannon wavelet is from π to 2π and it is 1 and from $-\pi$ to -2π this also is 1 and otherwise it is 0. So, that is the Fourier transform of the Shannon wavelet which means that my Shannon wavelet

$$\begin{aligned}
\psi(t) &= \mathscr{F}^{-1}\big[\hat{\psi}(w)\big] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iwt}\hat{\psi}(w)dw \\
&= \frac{1}{2\pi}\int_{-2\pi}^{-\pi} e^{iwt}dw + \int_{\pi}^{2\pi} e^{iwt}dw \\
&= \frac{1}{2\pi}\big[\sin(2\pi t) - \sin(\pi t)\big] \\
&= \frac{\sin(\pi t/2)\sin(3\pi t/2)}{\pi t/2}
\end{aligned}$$

So, after simplification, I see that the Shannon wavelet has the following form. So, if I were to plot these Shannon wavelets i.e. if I were to plot $\psi(t)$ versus t if I were to plot this function this function is an even function of t. Notice that the function is as given in figure. So, then all I have to see is whether this whether the Shannon wavelet and if I were to describe the discrete version of this Shannon wavelet, are they orthonormal or not. So, that is this question that I have to answer.

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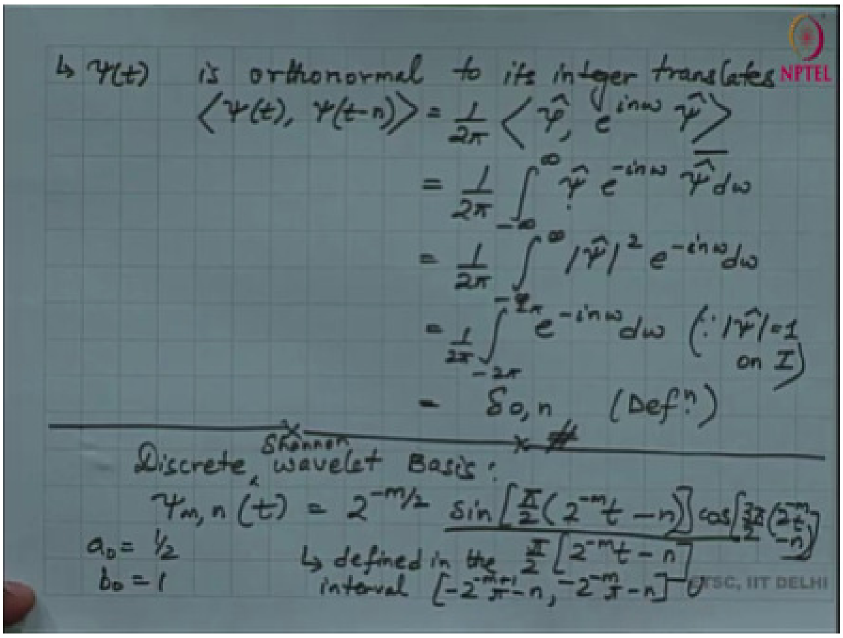

So, let us look at at $\psi(t)$. So, notice that $\psi(t)$ is orthonormal to its integer translates. So, orthonormal to its integer translates. So, what I mean to say is that

$$
\begin{aligned}
<\psi(t), \psi(t-n)> &= \frac{1}{2\pi} <\hat{\psi}, e^{inw}\hat{\psi}> \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\psi} e^{-inw}\overline{\hat{\psi}} dw \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{\psi}|^2 e^{-inw} dw \\
&= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} e^{-inw} dw \quad (\text{beacuse } |\hat{\psi}| = 1 \text{ on I}) \\
&= \delta_{0,n}
\end{aligned}
$$

Hence my result follows I have shown here is that ψ is orthonormal to its integer translates. So, then let us look at the corresponding discrete Shannon wavelet basis. So, I am going to again describe my discrete Shannon wavelet basis

$$
\psi_{m,n}(t) = 2^{-m/2} \frac{\sin\left[\frac{\pi}{2}(2^{-m}t - n)\right] \cos\left[\frac{3\pi}{2}(2^{-m}t - n)\right]}{\frac{\pi}{2}(2^{-m}t - n)}
$$

Here, $a_0 = 1/2$ and $b_0 = 1$ and $\psi_{m,n}(t)$ defined in the interval $\big[-2^{-m+1}\pi - n, -2^{-m}\pi - n\big]$

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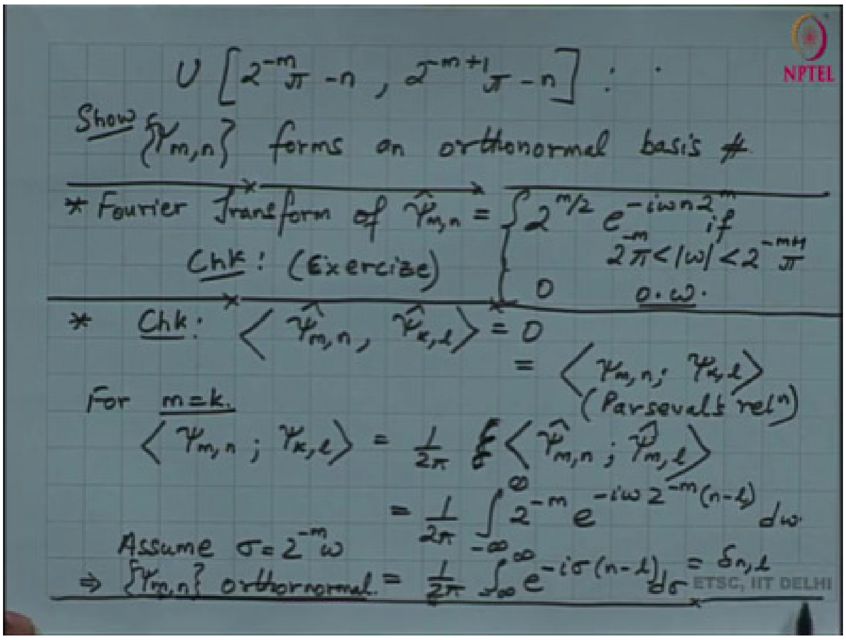

The other interval is $\cup\big[2^{-m}\pi - n, 2^{-m+1}\pi - n\big]$. So, then I have to show that this definition of $\psi_{m,n}$ forms and orthonormal basis. So, if that is the case, then the definition makes sense for the Shannon wavelet. So then let us look at the Fourier transform of $\hat{\psi}_{m,n}$

$$
\hat{\psi}_{m,n} = \begin{cases} 2^{m/2} e^{-iwn2^m}; & 2^{-m}\pi < |w| < 2^{-m+1}\pi \\ 0; & \text{otherwise} \end{cases}
$$

So, so I leave this as an exercise to the students to find that indeed the Fourier transform is the given function. So, this has already been done and I have already shown the result here. Further check the students are asked to check the following result that the $<\hat{\psi}_{m,n};\hat{\psi}_{k,l}>=0$. So, that can be easily done by taking the appropriate inner product of these Fourier transform of this discrete version of the functions.

$$<\hat{\psi}_{m,n};\hat{\psi}_{k,l}>=0=<\psi_{m,n};\psi_{k,l}> \quad \text{(Parseval's Rel.)}$$

For m=k;

$$<\hat{\psi}_{m,n};\hat{\psi}_{k,l}>=\frac{1}{2\pi}<\hat{\psi}_{m,n};\hat{\psi}_{m,l}>$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}2^{-m}e^{-iw2^{-m}(n-l)}dw$$

Notice that if I were to simplify this further, let me assume the following change of variables. So, assume my $\sigma = 2^{-m}w$. Then I have

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}2^{-m}e^{-i\sigma(n-l)}d\sigma=\delta_{n,l}$$

So, what I have shown here is that if m is equal to k then of course, these wavelets are orthonormal. So, I have shown here is that my psi m n my wavelets are orthonormal. So, for m not equal to k, students can check that the inner product will come out to be 0. So, these wavelets are orthonormal and hence the description of these discrete form of these wavelets make sense.

So, then in the next case, in this next example I am going to now introduce as another wavelet namely the Daubechies wavelets. Now Daubechies wavelets we will see that the special property of Daubechies wavelets is that we can describe how smooth or what sort of smoothness is the wavelet that we prescribe. So, the smoothness of the wavelet can be prescribed in this particular form of the wavelets. So, let us look at what is this construction.

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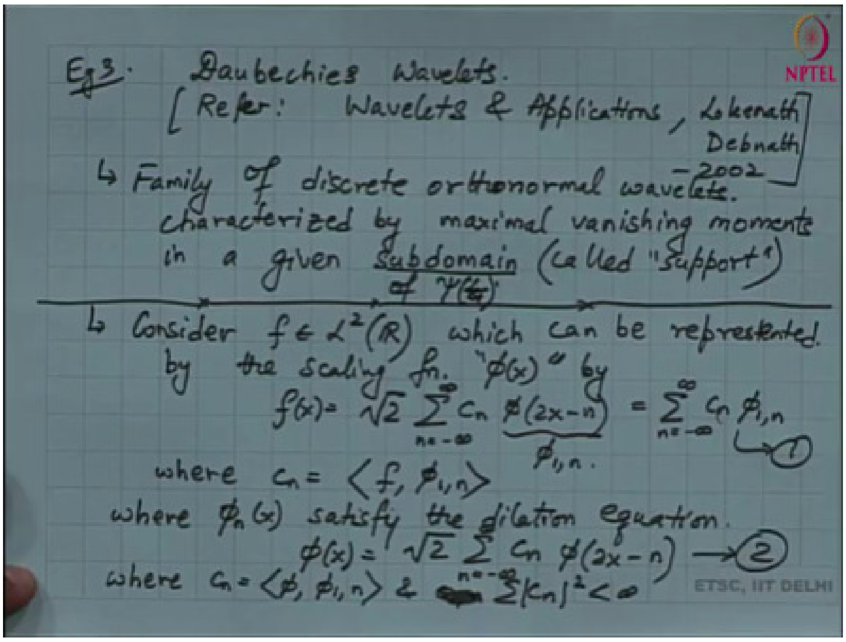

Now, before I move ahead, I would like to highlight to the students that for a very detailed description of Daubechies wavelets the students should refer the following text. Wavelets and Applications by Lokhnath Debnath, this was published in by Birkhauser in 2002. So, some of these ideas are taken from this references. So, let me just highlight now the Daubechies wavelet . So, first of all Daubechies wavelets are a family of discrete discrete orthonormal wavelets characterized by maximal vanishing moments. So, we can prescribe how many moments the wavelets should be set equal to 0 and that will describe how smooth that particular wavelet is. So, that is characterized by maximum vanishing moments in a given subdomain. So, maximum vanishing moments in a given subdomain of the wavelets ok.

I call these subdomains as support. So, for a given support if I can get maximal vanishing moments, then those particular class of wavelets are known as the debauchies wavelets. So, let us consider the construction of these wavelets first. So, let us consider a square integrable function f. So, now, I am going to represent this function f. So, the idea is as follows. From this function f, which is say any arbitrary square integrable function, I am going to represent this function in terms of certain scaling functions. Now we are going to look at some properties of the scaling functions namely, I am going to generate a particular function known as the generating function which will describe these wavelets. So, now let us consider these square integrable functions. So, these functions can be represented let us say that they are represented by the scaling function $\phi(x)$ by the following form:

$$f(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} C_n \ \phi(2x - n) = \sum_{n=-\infty}^{\infty} C_n \ \phi_{1,n}$$
$$\text{where,} \qquad C_n =< f, \phi_{1,n} >$$

Let me call this this definition of f by I. I am going to describe these scaling functions and I will use these definitions of scaling functions. So, that they are going to describe my Daubechies wavelets. So, let us call this we see that this these are my discrete form of these wavelets with $a_0 = b_0 = 1$. So, I can write rewrite this as above.

where, $\phi_n(x)$ satisfy the dilation equation

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} C_n \ \phi(2x - n)$$
$$\text{where,} \qquad C_n =< \phi, \phi_{1,n} > \quad \text{and} \quad \sum |C_n|^2 < \infty$$

Notice the idea is the same as what we have followed in the development of my discrete Haar wavelets where we had described my dilation function and using the dilation function I described the wavelet and hence the discrete wavelets. Let me call this dilation equation by II.

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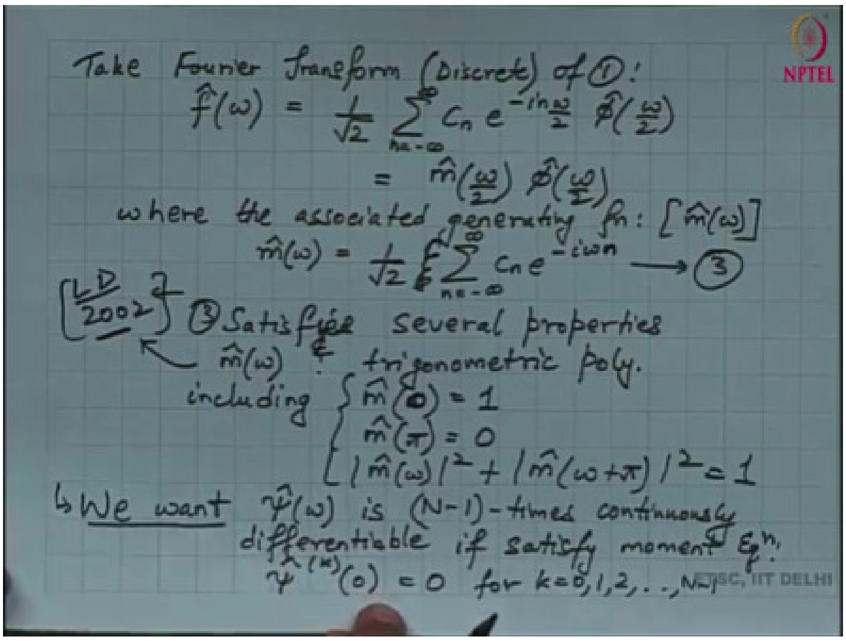

So, let us take let me now start taking the discrete Fourier transform of I.

$$\begin{aligned}\hat{f}(w) &= \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} C_n e^{-inw/2}\ \hat{\phi}(w/2)\\ &= \hat{m}(w/2)\ \hat{\phi}(w/2)\end{aligned}$$

where the associated generating function is:

$$\hat{m}(w) = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{\infty} C_n e^{-inw}$$

I am going to call this as III. Now, students asked again to refer to the texts which I have listed by Lokanath Debnath. Now mainly that there are certain properties of this associated generating functions again listed in this text. So, let me just highlight some of the properties that are satisfied by this associated generating function. So, I see that $\hat{m}(w)$ is a trigonometric polynomial. It is a trigonometric polynomial if I plug $w = 0$ and use the definitions of C_n described earlier $\hat{m}(0)$ is 1 and I also see that $\hat{m}(\pi)$ is 0. Further I see that $|\hat{m}(w)|^2 + |\hat{m}(w+\pi)|^2$ is equal to 1. Again all these relations are proved in detail in this particular text. So, I am going to use some of these results to give the construction of this particular Daubechies wavelet.

So, then let us continue looking at this expression III. We want to construct the Daubechies wavelet and to do that we want that $\hat{\psi}(w)$ is (n-1) times continuously differentiable. If now, this is the requirement that we want so that we are able to construct the Daubechies wavelet of the required vanishing moments. So, let us say there are n minus 1 vanishing moments and if that is the case we want that this psi hat of omega must satisfy the following moment must satisfy the following moment equation. It satisfies the moment equation given by psi hat k of 0 is 0. So, let us say, for k equal 0, 1, 2 to n minus 1 the derivative of this this wavelet at 0 is 0. And now so, if that is the case what we want is.