Integral Transform and Their Applications
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Lecture - 70
Discrete Haar, Shanon and Debauchies Wavelet- Part 01

Good afternoon everyone. So, we ended the last lecture on the discussion on discrete wavelet transforms, namely, we introduced discrete wavelet function using the definition of continuous wavelet functions. And towards the later half of the last lecture we saw that if this wavelets are orthonormal, then in that case the wavelet transforms for the discrete cases are well defined. So, today in this lecture, I am going to describe three different types of discrete wavelet transforms namely the Haar wavelets, the Shanon wavelets and the Daubechies wavelets. So, let us move on.

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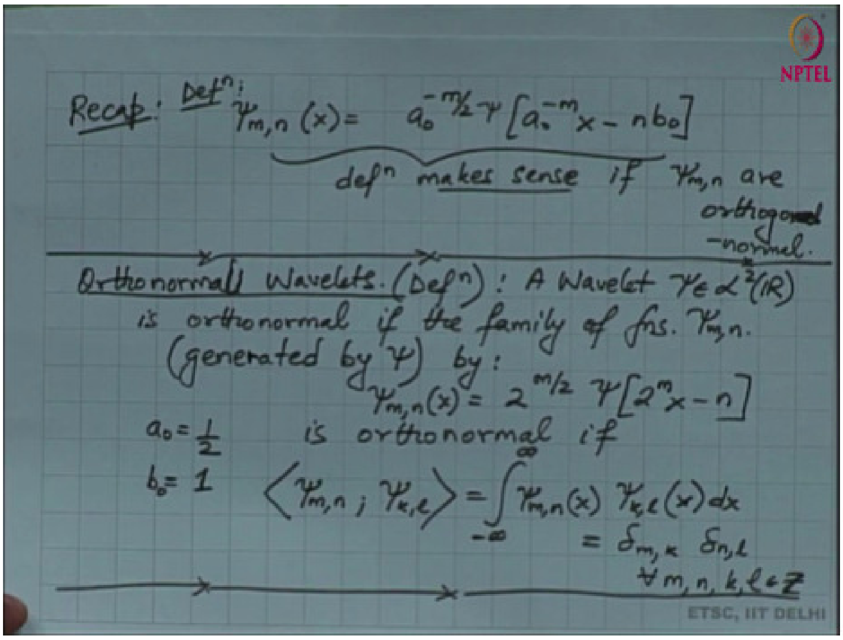

So, let me just again briefly recap what I just said. So, what I saw was:

$$\psi_{m,n}(x) = a_0^{-m/2}\psi\big[a_0^{-m}x - nb_0\big]$$

So, this definition makes sense, it makes sense if I have that my $\psi_{m,n}$ are orthonormal. So, that is the conclusion that we saw in my last lecture after the discussion on the definition of frames. So, if I can show that my discrete wavelets by these definitions are orthonormal then this particular definition from the continuous wavelets makes sense. So, whatever discrete wavelet I define all I need to make sure is that those particular wavelets are orthonormal.

So, let us look at some particular example here. So before I move ahead let me define what are orthonormal wavelets and how should I look for orthonormal wavelets. So a wavelet defined by all the square integrable function is orthonormal if the family of functions $\psi_{m,n}$ generated by the continuous wavelets ψ is given by the following definition

$$\psi_{m,n}(x) = 2^{m/2}\psi\big[2^m x - n\big]$$

Then corresponding to this continuous wavelet my discrete version of this wavelet will be orthonormal if I have the following:

$$\begin{aligned} <\psi_{m,n}, \psi_{k,l}> &= \int_{-\infty}^{\infty} \psi_{m,n}(x)\psi_{k,l}(x)dx \\ &= \delta_{m,k}\delta_{n,l} \qquad \text{for all} \quad m, n, k, l \in \mathbb{Z} \end{aligned}$$

So, if that happens then I have a family of discrete orthonormal wavelets and if that happens then my wavelets are well defined. So, all I have to worry about is find the orthonormality of the wavelets that I defined. Before I show some examples, let us look at a quick result. So, let me introduce a short result in the form of a lemma.

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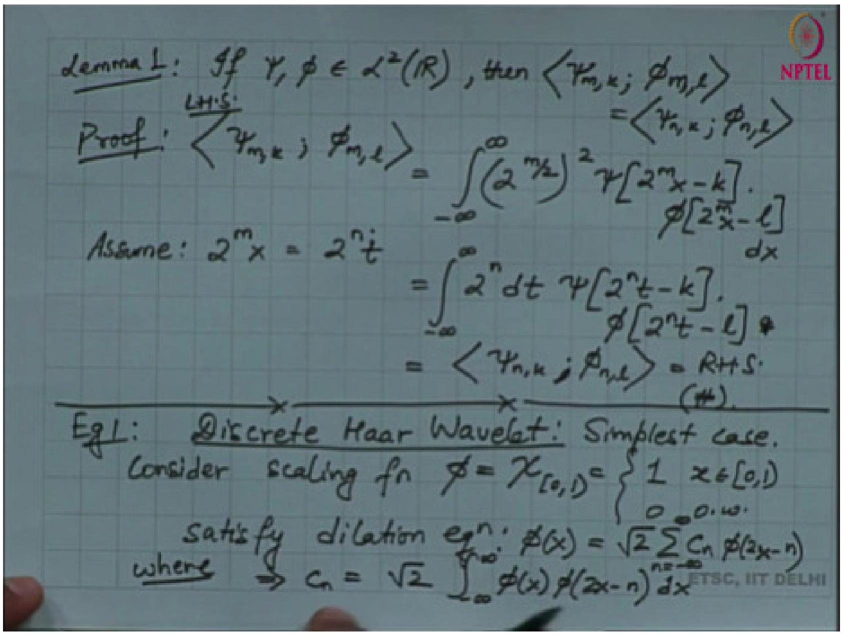

So, the result says that if ψ and ϕ are two square integrable functions, then

$$<\psi_{m,k};\phi_{m,l}> = <\psi_{n,k};\phi_{n,l}>$$

So, basically which means that we can replace the indices if ψ and ϕ are two square integrable functions. So, the proof is quite straight forward. Let us look at the left hand side of the statement:

$$<\psi_{m,k};\phi_{m,l}> = \int_{-\infty}^{\infty} (2^{m/2})^2\psi\big[2^m x - k\big]\ \phi\big[2^m x - l\big]dx$$

If I assume the index m is replaced by another index n, i.e. if I assume $2^m x = 2^n t$. Then,

$$\begin{aligned} &= \int_{-\infty}^{\infty} 2^n dt\ \psi\big[2^n t - k\big]\ \phi\big[2^n t - l\big] \\ &= <\psi_{n,k};\phi_{n,l}> = \text{R.H.S.} \end{aligned}$$

So, then let us take the discrete haar wavelet case. As we have shown here the continuous version is a very simple function, even the discrete version is the simplest of the discrete case that we can look at the simplest case. Let us now consider the following continuous function. So, I am going to call this as my scaling function.

$$\phi = \chi_{\big[0,1\big)} = \begin{cases} 1, & x \in \big[0,1\big) \\ 0, & \text{otherwise} \end{cases}$$

So, my characteristic function is defined as above. Notice that this particular characteristic function is by definition is going to satisfy the following equation or the dilation equation. So, what I have is that:

$$\phi(x) = \sqrt{2} \sum_{n=-\infty}^{\infty} C_n \phi(2x - n)$$
$$\text{where,} \quad C_n = \sqrt{2} \int_{-\infty}^{\infty} \phi(x)\ \phi(2x-n) dx$$

So, that is how my coefficients are calculated. Now the fact that ϕ is just a characteristic function over this interval [0,1), I can readily calculate this coefficients So, what I get is the following.

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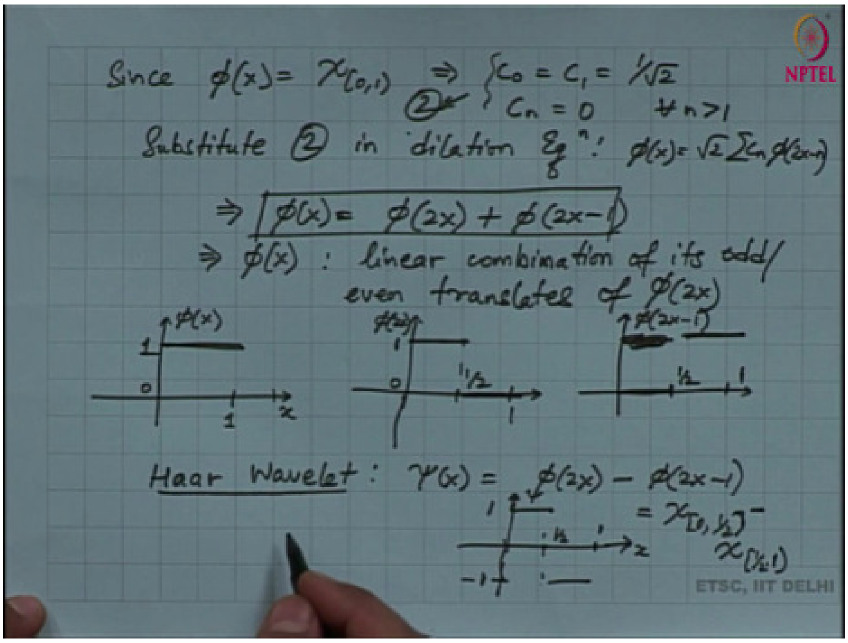

Since

$$\phi(x) = \chi_{\big[0,1\big)} => \begin{cases} C_0 = C_1 = 1/\sqrt{2} \\ C_n = 0, \quad \text{for all} \ \ n > 1 \end{cases}$$

Let us call this as II, if I were to substitute II in this expression in my dilation equation, which is $\phi(x) = \sqrt{2} \sum C_n\ \phi(2x - n)$. So if I were to substitute all these values of C in this dilation equation, I see that I get the following result.

$$\phi(x) = \phi(2x) + \phi(2x - 1)$$

I have shown here that the ϕ is the linear combination of its odd and even translates of $\phi(2x)$. Now, if you were to draw these three functions, I can see them in the figure.

Now, if I were to describe my haar wavelet $\psi(x)$ then that would be nothing but, $\phi(2x)+\phi(2x-1)$. Why? Because I know my haar wavelet is 1 from 0 to half and it is -1 from half to 1(for the other half of the interval). So, that is my haar wavelet in terms of the scaling functions. So, then I can express this in terms of the characteristic function, which is $\chi_{\left[0,1/2\right)} - \chi_{\left[1/2,1\right)}$. In terms of characteristic function that is my continuous version of the haar wavelet. So, now let us look at the discrete version of this wavelet.

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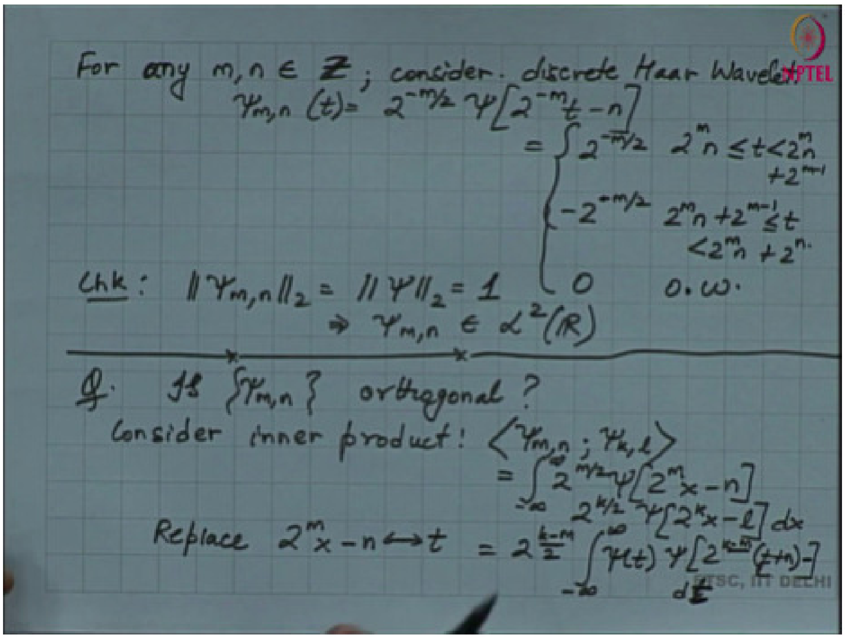

So, for any integer say for any m,n which is in $\mathbb{Z}$, let us consider the discrete the discrete haar wavelet So, let us start constructing the discrete haar wavelet.

$$\psi_{m,n}(t) = 2^{-m/2}\ \psi\left[2^{-m}t-n\right] = \begin{cases} 2^{-m/2}; & 2^m n \leq t < 2^m n + 2^{n-1} \\ -2^{-m/2}; & 2^m n + 2^{m-1} \leq t < 2^m n + 2^n \\ 0; & \text{o.w.} \end{cases}$$

So, this definition of the discrete version of the haar wavelet follows from the regular continuous version that we are familiar. So, students are requested to check that the L-2 norm of this discrete haar wavelet is the same as the L-2 norm of the continuous haar wavelet and also requested to check that this norm is equal to 1. So, which means that the discrete haar wavelet is a square integrable function.

Next I have to see whether these wavelets are orthogonal or not? I have already shown you that this wavelets have a norm of 1. So, that is left as an exercise to the students, so the question is: Are these discrete wavelets orthogonal or not? Otherwise the definition may not make sense. So, let us now consider the inner product of these discrete wavelets.

$$<\psi_{m,n};\psi_{k,l}>=\int_{-\infty}^{\infty} 2^{m/2}\ \psi\left[2^m x-n\right]2^{k/2}\ \psi\left[2^k x-l\right]dx$$

So, then if I were to use the same replacement that we did for our lemma, i.e.,

$$2^m x - n \longleftrightarrow t;$$

$$\text{we get,} \qquad 2^{(k-m)/2} \int_{-\infty}^{\infty} \psi(t)\ \psi\big[2^{(k-m)}(t+m-l)dt\big]$$

So, then this integral can now be looked for two different cases. First case could be when $k = m$ and in that case I will show that this integral reduces to 1. And in the second case if $k \neq m$ I will show that the integral reduces to 0 and hence we will show that these inner product of these two wavelets are orthonormal. So, let us look at the two cases.

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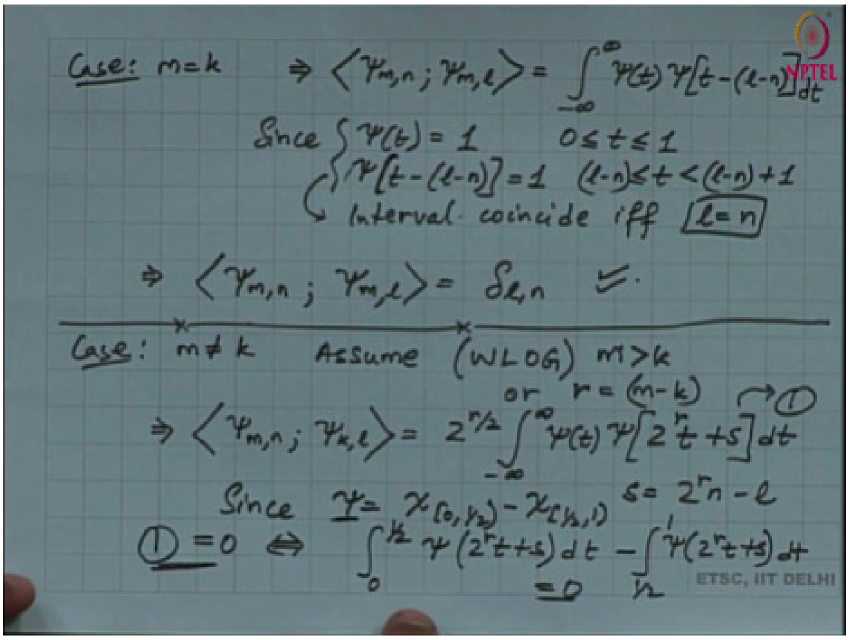

So, the first case is when I have $m = k$;

$$< \psi_{m,n}; \psi_{m,l} >= \int_{-\infty}^{\infty} \psi(t)\ \psi\big[t-(l-n)\big]dt$$

Notice that since I have that $\psi(t)$ is already a haar wavelet. So,

$$\begin{cases} \psi(t) = 1; & 0 \leq t \leq 1 \\ \psi\big[t-(l-n)\big] = 1; & (l-n) \leq t < (l-n)+1 \end{cases}$$

So, I see that due the way we have defined the function ψ, they take values equal to 1 at these respective intervals. And notice that these two intervals coincide if and only if I have l is equal to n,which means

$$< \psi_{m,n}; \psi_{m,l} >= \delta_{l,n}$$

Now let us look at the case when k is not equal to m. We could either have k less than m or k greater than m. But let us now assume without loss of generality that my m is greater than k. We can prove the other case in a similar fashion to this case. So, I leave that case to the students as an exercise. So, let us now say that I have that $r = m - k$. So, then let us consider this inner product.

$$< \psi_{m,n}; \psi_{k,l} >= 2^{r/2} \int_{-\infty}^{\infty} \psi(t)\ \psi\big[2^r t + s\big] dt$$

I have introduced a new variable $s = 2^r n - l$. So, I have simplified my inner product using this new variable S. So, notice that since psi is my haar wavelet; so psi is my haar wavelet. So, psi is the following characteristic function.

$$\psi = \chi_{\big[0,1/2\big)} - \chi_{\big[1/2,1\big)}$$

Let me call this let me call this integral as I. So, I is identical is equal to 0, if and only if we show that the following integral:

$$\int_0^{1/2} \psi(2^r t + s) dt - \int_{1/2}^{1} \psi(2^r t + s) dt = 0$$

Now, let us look at these two particular integrals.

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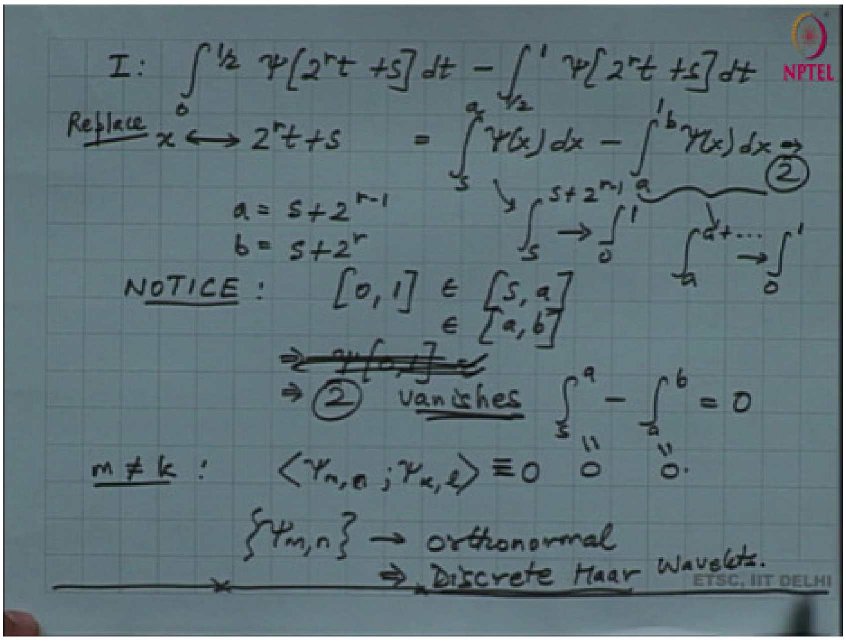

$$\text{I} : \int_0^{1/2} \psi(2^r t + s) dt - \int_{1/2}^{1} \psi(2^r t + s) dt$$

So notice that if I were to replace x by $2^r t + s$, I see that:

$$= \int_s^a \psi(x) dx - \int_a^b \psi(x) dx$$

where, $a = s + 2^{r-1}, b = s + 2^r$. So, all I have to show is that these two integrals and the difference of these two integrals over these limit is 0. So, but notice the following that my interval 0 1; it lies both in this limit S a and it also lies in the limit in the limit a b. Why? Because notice that this particular integral is integral from S to S plus some value 2 to the power r minus 1. So, I can appropriately shift this integral from 0 to 1; shift and scale So, we can shift and scale this integral so that it becomes an integral from 0 to 1. And similarly notice that this particular integral is integral from a to a; well it will be a to a plus a plus some factor which we can evaluate and again this integral can also be shifted to 0 to 1 ok. So, notice that 0 to 1 is a support or a sub domain of both these interval and I know; I know that that psi in this in this interval will well I know what is the definition of psi in this interval; so in this interval 0 to 1; let me call this is integral as my 2 in this. So, so what I have is the conclusion that 2 is going to; 2 vanishes that is this integral from S to a minus the integral from a to b is 0. Because it can be appropriately shifted and scaled to have it from have the integral from 0 to 1. And from there I can show that due to the definition of psi the integrals individually; individually they are 0 ok. So, so what have I shown here is if m is not equal to k; then the inner product of m, well m l the inner product of m n with psi of k; l it is identically 0. Or hence I have shown that psi m n; the way we have described my discrete transform, this is orthonormal. So, this is orthonormal and hence this defines nicely defines the discrete haar wavelet basis ok. So, so that completes my discussion on the description of haar discrete haar wavelets ok. So, all we need to see is whether those wavelets are orthonormal.