Integral Transform and Their Applications
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Lecture - 66
Fractional ODEs and PDEs (Continued) - Part 3

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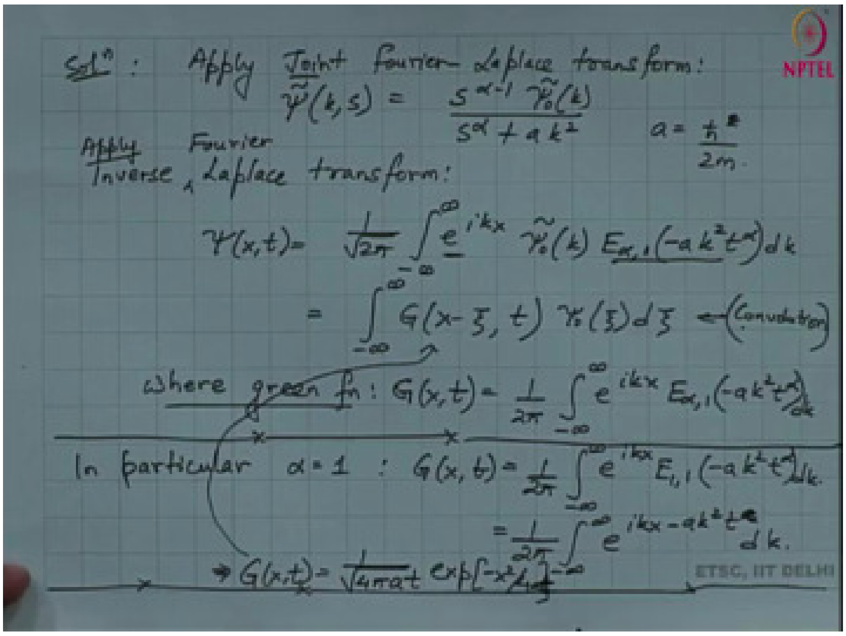

So, let us move on to see a more general case of a PDE which involves examples including Burgers equation for Shock mechanics or the telegraph equation or the KdV equation for water waves or the wave equation or the diffusion equation. So, I am going to look at the solution of the most general constant coefficient PDE using my Fourier joint Fourier Laplace transform. So, let us continue a discussion.

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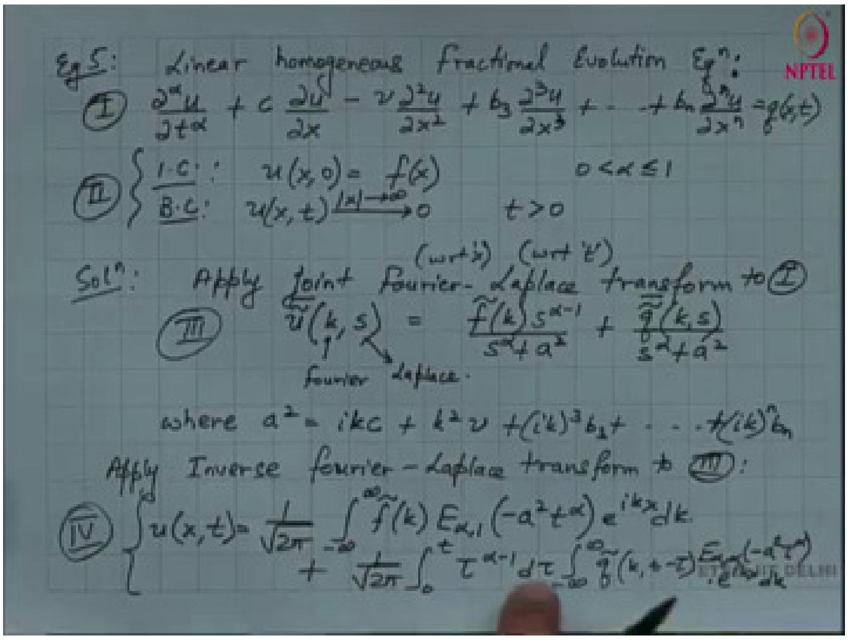

So, what I have is the following. Let me call the PDE as linear homogeneous fractional evolution equation. So, the equation is as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + c\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} + b_3\frac{\partial^3 u}{\partial x^3} + \cdots + b_n\frac{\partial^n u}{\partial x^n} = q(x,t)$$

So, let me try to solve, let me give all the initial and boundary conditions also:

$$u(x,0) = f(x); \qquad 0 < \alpha \leq 1$$
$$u(x,t) \longrightarrow 0 \quad \text{as} \quad |x| \to \infty; \qquad t > 0$$

Let me call this my expression I and this conditions as my II. Let me further assume that now fractional order is from 0 to 1. So, note that if I were to take $c \neq 0$ and all the other coefficients ν, b_3, b_4, b_n to be 0, I get my advection equation or if I take $c = 0$ and $\nu \neq 0$ and all the other coefficient are 0, I get my diffusion equation and if I take $b_3 \neq 0$, I get my KdV equation. So, all the equations are enveloped in this joint PDE. So, let me just try to solve this PDE and give the most general solution to this constant coefficient PDE.

So, let me just start by right away applying my joint Fourier Laplace transform,

$$\bar{\tilde{u}}(k,s) = \frac{\tilde{f}(k)s^{\alpha-1}}{s^\alpha + a^2} + \frac{\bar{\tilde{q}}(k,s)}{s^\alpha + a^2}$$

$$\text{where, } a^2 = ikc + k^2\nu + (ik)^3 b_3 + \cdots + (ik)^n b_n$$

Let me call this as my expression III. We see that a square envelops all the different types of special derivatives that are included in this joint PDE which means that, I can always invert my expression III to come back to my original solution to the PDE. So, if I were to apply inverse Fourier Laplace transform to III, I get the following expression:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(k)E_{\alpha,1}(-a^2t^\alpha)e^{ikx}dk + \frac{1}{\sqrt{2\pi}}\int_0^t \tau^{\alpha-1}d\tau\int_{-\infty}^{\infty} \tilde{q}(k,t-\tau)E_{\alpha,\alpha}(-a^2t^\alpha)e^{ikx}dk$$

So, this further evaluation of the solution is not possible at this stage, because we are not given the form of f and q, however I just want to give you the following remark this particular equation encompasses lots of different types of PDEs.

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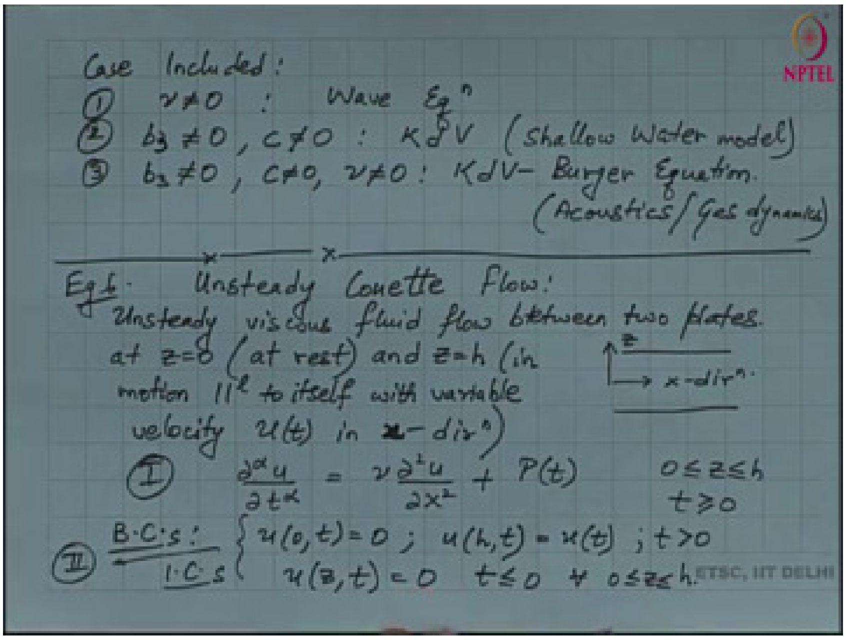

Namely the cases that are included in this example. So, case are included as, if I am given that $\nu \neq 0$ there is no diffusive term then I am dealing with the wave equation and if I have the coefficient of the third derivative $b_3 \neq 0$ and $c \neq 0$, then I have the famous fractional KdV equation for shallow waters or shallow water models and then if I have that $b_3 \neq 0, c \neq 0, \nu \neq 0$, I have the KdV burgers equation which models the shock in a fluid. So, this is for acoustics and gas dynamics. So, then let us look at another example.

So, the second example will be for another fluid flow case that is the unsteady Couette flow. So, let me just briefly highlight what is this unsteady Couette flow. Suppose we are given a pipe in two dimensions or we can assume that there are 2 plates in between which that there is a fluid flow.

So, such flow where we have such bounded flow between two plates is said to be a Couette flow. Basically this is a flow in a finite domain. So, we have a fluid flow in a in a finite domain here. So, let me write down the type of problem we are solving here. So, this is unsteady viscous fluid flow between two plates at $z = 0$. So, let me call that the plate at $z = 0$ is at rest and the second plate $z = h$ is in motion which is parallel to itself with variable velocity $u(t)$ in x direction.

So, let me just set my frame of reference, my x direction and my z direction is perpendicular to this. So, this is my z direction. So, the equation that we are solving for the fluid flow in this case is the fractional diffusion equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^2 u}{\partial x^2} + P(t); \quad 0 \le z \le h, \quad t > 0$$

Let me call this equation as I. So, I am given further I am given the boundary and initial condition are as follows:

$$u(0,t) = 0; \quad u(h,t) = u(t); \quad t > 0$$
$$u(z,t) = 0; \quad t \le 0, \quad \text{for all } \; 0 \le z \le h$$

Let me call these initial and boundary conditions as II. So, solving I and II let us look at the solution.

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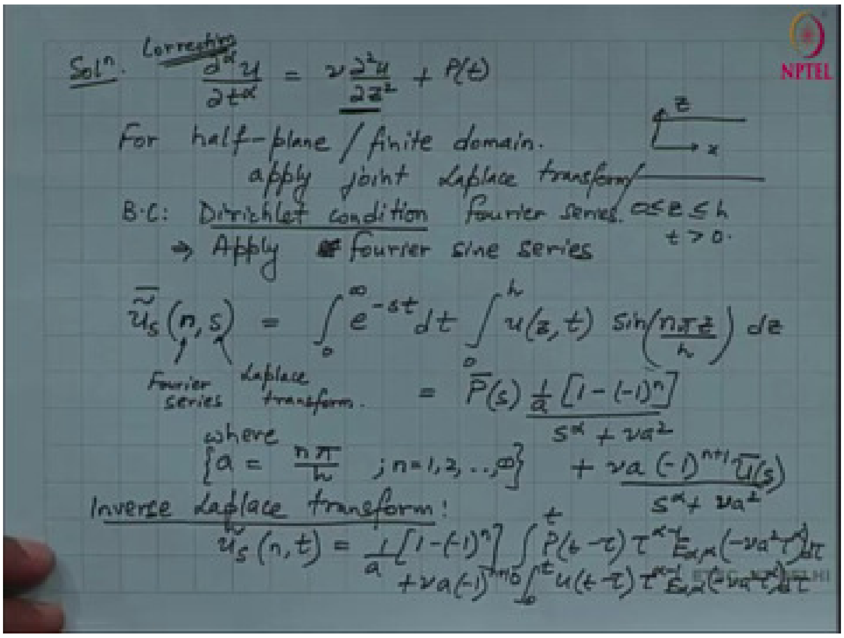

So now, why I brought up this particular example is the following? Since we are solving in a finite domain I cannot apply a domain is as follows 0 to h and $t > 0$. We see that my domain in the z direction is finite and my PDE is with respect to the z derivative. So, I need to correct. Thus,

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^2 u}{\partial z^2} + P(t); \quad 0 \leq z \leq h, \quad t > 0$$

So, the derivative is with respect to z. So, which means our regular application of Fourier transform or Laplace transform is not going to work and we may have to resort to Fourier series because for finite domain we can use the so called periodic boundary condition and that is where my Fourier series is going to play a role in the solution to this PDE. So, for half plane or finite domains for half planes or finite domains, this case we have to apply joint Laplace transform, but Fourier series Laplace transform, but Fourier series.

So, what I see is now notice that I am also given my boundary condition as a Dirichlet condition. So, since boundary conditions are Dirichlet conditions because we are given the solution on the boundary. So, whenever we are given Dirichlet condition, we apply Fourier sine series and the typical method is if you are given Neumann condition apply Fourier cosine series. So, then if you do that, my joint Fourier Laplace solution is as follows where instead of a k where $-\infty < k < \infty$, I now have n. So, this is my Fourier series variable transformed variable and s corresponds to time is my Laplace transform variable. So, then I have:

$$\bar{\tilde{u}}_s(n, s) = \int_0^\infty e^{-st} dt \int_0^h u(z, t) \sin\left(\frac{n\pi z}{h}\right) dz$$

So, notice that the application of the sine transform brings in this factor $\sin\left(\frac{n\pi z}{h}\right)$. So, then when I apply my sine series, I can evaluate this interior integral and after evaluation gives me the following expression:

$$= \bar{P}(s)\frac{1/a(1-(-1)^n)}{s^\alpha + \nu a^2} + \frac{\nu a(-1)^{n+1}\bar{u}(s)}{s^\alpha + \nu a^2}$$

$$\text{where, } a = \frac{n\pi}{h}; n = 1, 2, \cdots, \infty$$

So, I have already applied Fourier and Laplace transform. So, let us now invert the solution. So, let us invert by applying the inverse Laplace transform and I get that my solution in the Fourier space:

$$\tilde{u_s}(n,t) = \frac{1}{a}\left[1-(-1)^n\right]\int_0^t P(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(-\nu a^2\tau^\alpha)d\tau + \nu a(-1)^{n+1}\int_0^t u(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(-\nu a^2\tau^\alpha)d\tau$$

Let me call this as my expression number III.

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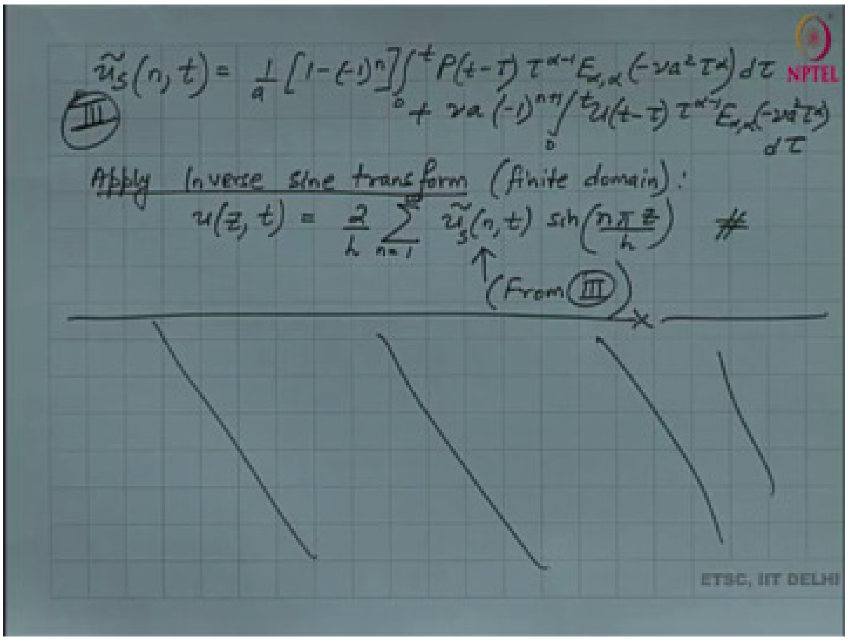

Now then when I apply my inverse sine transform and this time on a finite domain instead of the transform being the integral transform this is going to be a Fourier sine series. So after applying the inverse sine transform as the Fourier series, I get my original solution as follows:

$$u(z,t) = \frac{2}{h}\sum_{n=1}^{\infty}\bar{u}_s(n,t)\sin\left(\frac{n\pi z}{h}\right)$$

So, where this particular transform is coming from expression III. So, after substituting the solution from expression III, I get back my solution in the physical plane by summing up this particular series. So, that completes the solution to this example and also our discussion on fractional calculus and fractional PDEs. Now in my next set of 2 lectures, I am going to come and discuss on a new transform which has revolutionized almost entire engineering and physics. So, starting from bio medical engineering to signal processing to physics to science, you name the area and this particular transform is always there. And that particular transform is the wavelet transform. So, thank you for listening today and we will meet next time. Thank you!