Integral Transform and Their Applications
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Lecture - 65
Fractional ODEs and PDEs (Continued) - Part 2

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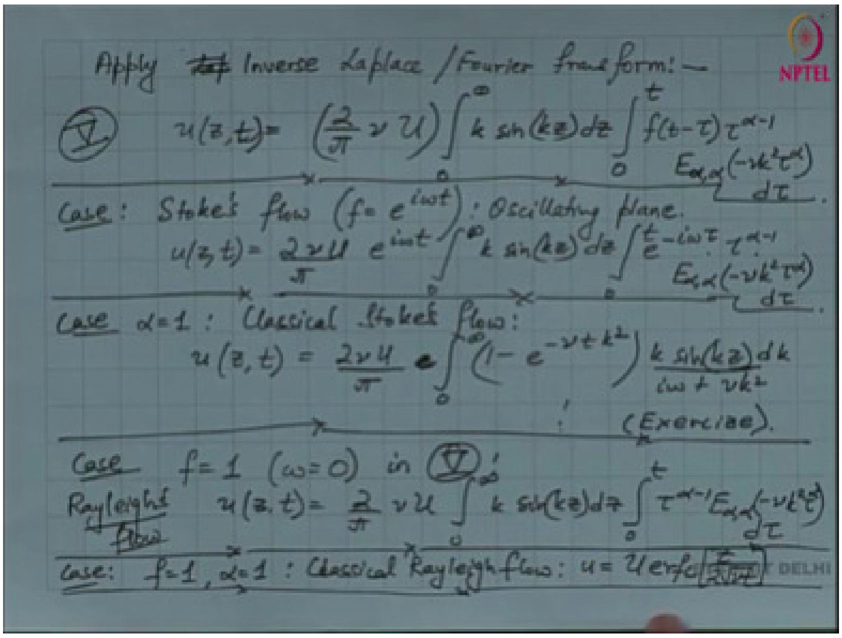

So, let me just talk about the case which is the Stokes flow. I am given that f is an oscillatory force, so if I am given that the boundary condition is an oscillatory boundary condition. So, this is the case of oscillating plate. So, if I plug $f = e^{i\omega t}$ I get:

$$u(z,t) = \left(\frac{2}{\pi}\nu u\right) e^{i\omega t} \int_0^\infty k \sin(kz) dz \int_0^t e^{-i\omega\tau} \tau^{\alpha-1} E_{\alpha,\alpha}(-\nu k^2 \tau^\alpha) d\tau$$

So this integral can be left here. Now there are some other special cases I can see. In particular let me look at a case where $\alpha = 1$. So, that is the classical Stokes flow. If I were to look at this case I plug $\alpha = 1$ in my expression, I see that the Leffler expansion is going to reduce to a pure exponential and I see that the solution in this case looks as follows:

$$u(z,t) = \left(\frac{2}{\pi}\nu u\right) \int_0^\infty (1 - e^{-\nu t k^2}) \frac{k \sin(kz)}{i\omega + \nu k^2}$$

So, further evaluation is left to the students as an exercise. So, please try to evaluate this integral further by using integration by parts, this equation can be further solved. So, then the next steps I leave it as an exercise.

Then I can talk another set of two cases. Let me talk about a case which is the Rayleigh flow that is when f is given to be 1 or ω is 0. So, let me call the solution from previous lecture as my expression number V. So, if I plug $\omega = 0$ or $f = 1$ in in in my expression V, I get the Rayleigh?s flow. So, in Rayleigh?s flow the solution is:

$$u(z,t) = \left(\frac{2}{\pi}\nu u\right)\int_0^\infty k\sin(kz)dz\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-\nu k^2\tau^\alpha)d\tau$$

So, further again, further evaluation of the expression needs a bit more knowledge about how to evaluate this interior integral, and it is left at this point. So this particular case is the Rayleigh?s flow, where the plate is held stationary, so Rayleigh flow. The fourth case that we can look at is the classical Rayleigh?s flow. So, case when $f = 1$ and $\alpha = 1$. So, this is the classical Rayleigh?s flow and I need to plug $f = 1$ and $\alpha = 1$ and I get back my classical Rayleigh solution as follows:

$$u = U\ erfc\left[\frac{z}{2\sqrt{\nu t}}\right]$$

So, these were some of the cases for Rayleigh?s and Stokes flow that were discussed in this example. So, in my next example I am going to look at a wave equation and then followed by I am going to look at some Schrodinger equation and in fact, I am going to show you an example where I see a most general fractional PDE.

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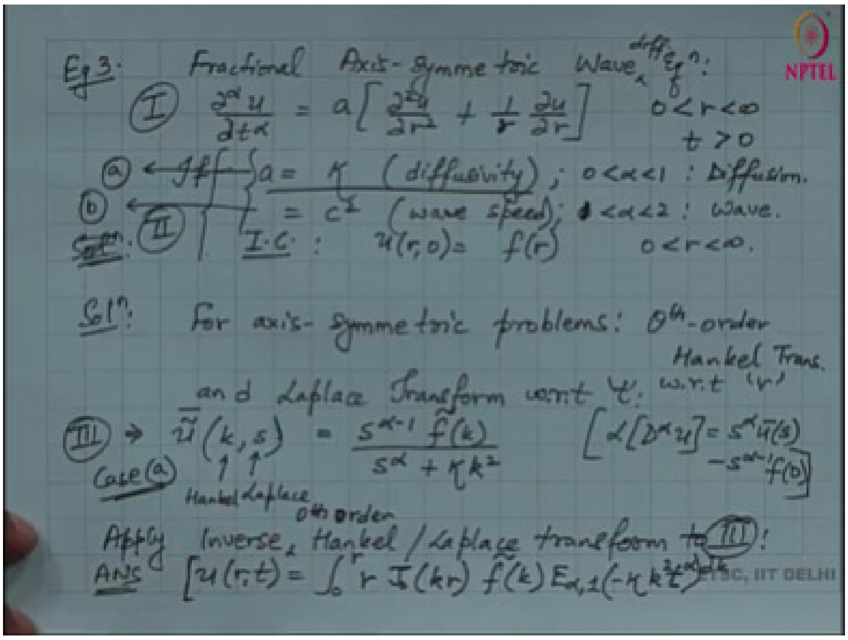

So, let us start with the wave equation case. So, in this example we will solve the fractional axis symmetric wave diffusion equation. So, the equation is as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]; \qquad 0 < r < \infty, \quad t > 0$$

I am going to call this PDE by I and then let me just highlight the fact that this particular equation can be broken down into two separate types of equations, namely if I were to look at the case where I take my $a = \kappa$ that is the diffusivity, then I am talking about and further $0 < \alpha < 1$, I am talking about the diffusion equation. So, that is the case of diffusion. Furthur if I choose my $\alpha = c^2$, where c is the wave speed and I choose my α like $1 < \alpha < 2$, I have the so called wave equation. So, notice that if I choose my α to be integral integer values in both these cases, we see that this is the diffusion and the wave equation and we have seen in our discussion on Hankel transforms that for axis symmetric problems the best way to solve these problems are via Bessel function. So, let us look at the solution to this problem. Before that let me also provide you with the initial condition.

$$u(r,0) = f(r); \qquad 0 < r < \infty$$

Let me call this initial condition by II. So, then as I said for axis symmetric problem let us apply the 0th order Hankel transform. Well, the boundary condition is trivial the solution needs to be bounded. So, if I were to apply the 0th order Hankel transform and that is transformed with respect to the variable r and Laplace transform with respect to my variable t, I get the joint Hankel for Laplace transform as follows:

$$\bar{\tilde{u}}(k,s) = \frac{s^{\alpha-1}\tilde{f}(k)}{s^{\alpha} + \kappa k^2}$$

Here I have directly applied my initial condition. Here we see that the right-hand side is the Laplacian. So, applying the Hankel transform will bring in k^2 and we are solving the heat equation hence this diffusivity constant and then the next step is to invert this expression.
Let me call this as expression III and I have used the following application of Laplace transform:

$$\mathscr{L}\left[D^{\alpha}u\right] = s^{\alpha}\bar{u}(s) - s^{\alpha-1}f(0)$$

So, then let us apply the inverse Hankel and Laplace transform to transform to my expression given by III, I get to this following point. I see that my solution is as follows:

$$u(r,t) = \int_0^r rJ_0(kr)\tilde{f}(k)E_{\alpha,1}(-\kappa k^2 t^{\alpha})dk$$

So, that brings in the solution. So, the moment we are able to solve this integral that gives us the solution to this axis symmetric diffusion equation. So, this is the point where we stop finding the further answer to this expression. So far I have shown you case I here. So, let me call this diffusion equation as my case 'a' and let me call this wave equation as my case 'b'. So, far what I have done is I have shown this is for case 'a' here. This is for diffusion equation. So, for case 'b', I need to look at the case where $1 < \alpha < 2$, which means I will have two initial conditions for the axis symmetric wave equation.

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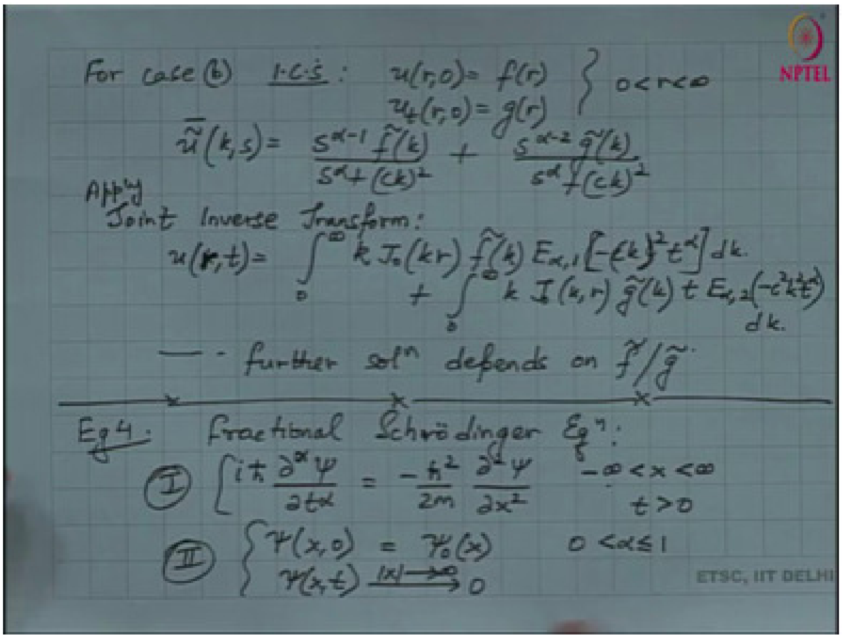

For case 'b' let me say that my initial conditions are as follows:

$$u(r,0) = f(r)$$
$$u_t(r,0) = g(r); \qquad 0 < r < \infty$$

Then my transformed variable $\tilde{\bar{u}}$ is as follows:

$$\tilde{\bar{u}}(k,s) = \frac{s^{\alpha-1}\tilde{f}(k)}{s^\alpha + (ck)^2} + \frac{s^{\alpha-2}\tilde{g}(k)}{s^\alpha + (ck)^2}$$

So, then then if I were to apply joint inverse transform right away. Then,

$$u(r,t) = \int_0^\infty kJ_0(kr)\tilde{f}(k)E_{\alpha,1}\big[-(ck)^2t^\alpha\big]dk + \int_0^\infty kJ_0(u,r)\tilde{g}(k)tE_{\alpha,2}\big[-(ck)^2t^\alpha\big]dk$$

Then the further solution depends on the functional form of f and g depends on $\tilde{f}$ and $\tilde{g}$. So, this expression can be further solved based on the functional form of f and g, which is left at this point.

So, next I am going to give you and show you another example of fractional PDE in quantum mechanics. So, that is the fractional Schrodinger equation. So, what I have is the equation to be solved:

$$i\hbar\frac{\partial^\alpha\psi}{\partial t^\alpha} = \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}; \qquad -\infty < x < \infty;\ t > 0$$

Let me call this as $I$and I see that the initial and boundary condition is :

$$\psi(x,0) = \psi_0(x); \qquad 0 < \alpha \leq 1$$
$$\psi(x,t) \longrightarrow 0 \quad \text{as} \quad |x| \to \infty$$

Let me call these initial and boundary condition as II. So, if $\alpha = 1$, I get the regular Schrodinger equation. So, I am going to use Fourier transform in space and Laplace transform in time. So, let me just jointly use Fourier Laplace.

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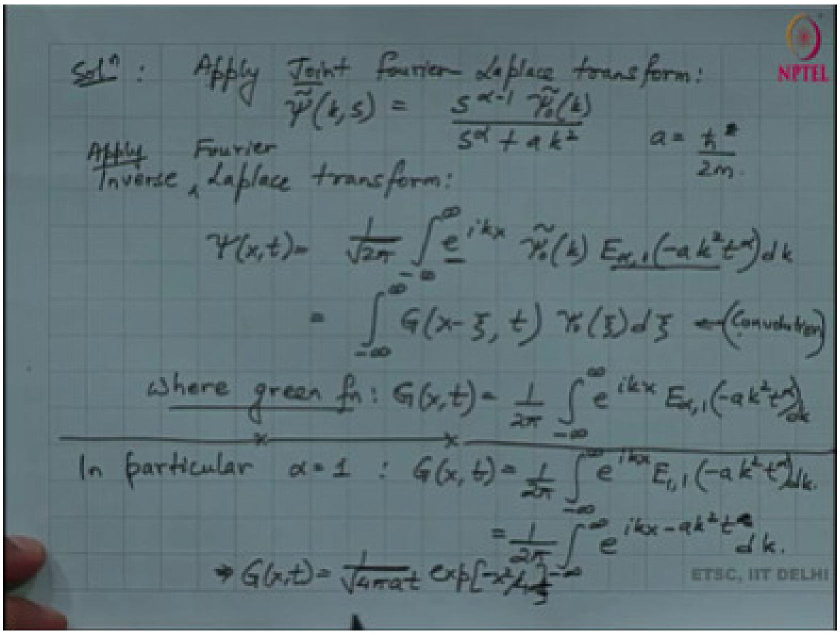

So, to find the solution in the transformed plane let us apply joint Fourier Laplace transform, we get:

$$\bar{\tilde{\psi}}(k,s) = \frac{s^{\alpha-1}\tilde{\psi}_0(k)}{s^\alpha + ak^2}, \quad \text{where,} \quad a = \frac{\hbar}{2m}$$

So, well before I move ahead, I just want to highlight that so far we have done so many examples involving fractional ODEs and PDEs that I believe that students are able to simplify and come to this point of the expression by applying joint Fourier and Laplace transform.

So, some of the steps here come applying joint Fourier Laplace transform and coming to this point has been skipped based on the fact that the students can see those steps in some of my previous example. So, right away I write the solution in the transformed plane and then the next step is to apply the inverse transform of the solution. So, let us apply both inverse Fourier and inverse Laplace simultaneously. So, when I do that I get that my wave function in my physical plane is as follows:

$$\begin{aligned}\psi(x,t) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{ikx}\tilde{\psi}_0(k)E_{\alpha,1}(-ak^2t^\alpha)dk \\ &= \int_{-\infty}^\infty G(x-\zeta,t)\psi_0(\zeta)d\zeta\end{aligned}$$

So, this is my solution in terms of convolution, where my Green?s function is as follows:

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}E_{\alpha,1}(-ak^2t^\alpha)dk$$

So, then there is one more step that can be done to come to a slightly more simplified version of the solution. In particular I can look at the case where $\alpha = 1$. Then,

$$\begin{aligned}G(x,t) &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}E_{1,1}(-ak^2t^\alpha)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx-ak^2t}dk\end{aligned}$$

So, I can write away find this particular integral and I get that my Green?s function is as follows:

$$G(x,t) = \frac{1}{\sqrt{4\pi a t}}\exp\left[\frac{-x^2}{4at}\right]$$

So, those are my Green?s function for the case when I am working for the classical Schrodinger equation and then that can be plugged back into my convolution integral. So that completes my discussion on the Schrodinger equation here.