Integral Transform and Their Applications
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Lecture - 64
Fractional ODEs and PDEs (Continued) - Part 1

Good afternoon everyone. So, in today?s lecture I am going to continue my discussion on Fractional PDE?s and in today?s lecture we are going to give you lots of examples of fractional PDE?s arising in fluids, in mechanics, in quantum mechanics, in signal processing, in waves, in diffusion, in porous media. And today?s lecture will be one of the last lectures on the discussion on fractional calculus. So, moving on let me right away start with an example.

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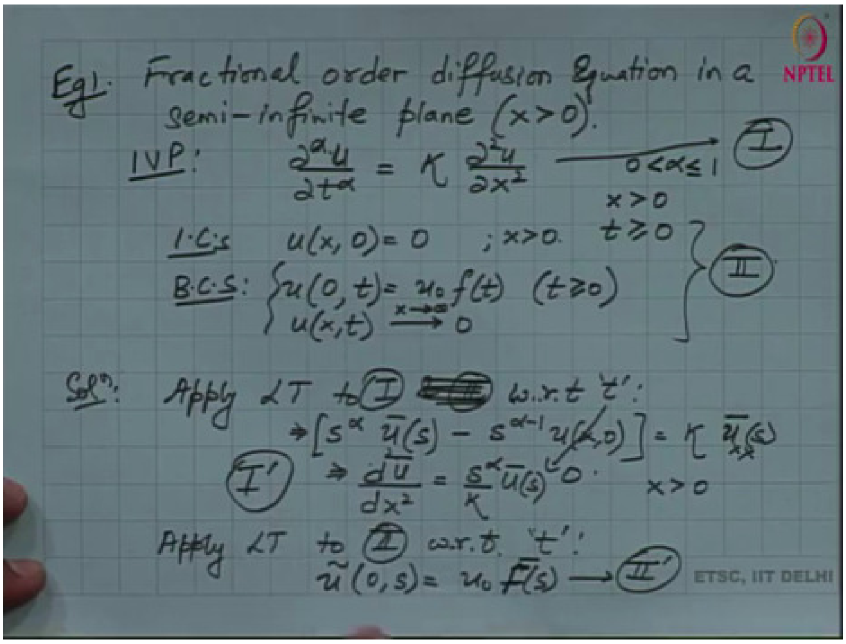

So, the example that I have in mind is the fractional diffusion equation on the half plane. In this question, we have to solve fractional order diffusion equation in a semi infinite domain ($x > 0$). So, this is an initial value problem given by the following PDE:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \kappa \frac{\partial^2 u}{\partial x^2}; \qquad 0 < \alpha \leq 1,\ x > 0,\ t \geq 0$$

Then my initial conditions are as follows:

$$u(x, 0) = 0; \qquad x > 0,\ t \geq 0$$

and the second I need two sets of boundary conditions for this second order PDE in space and those are the:

$$u(0, t) = u_0 f(t); \qquad t \geq 0$$
$$u(x, t) \longrightarrow 0 \qquad \text{as} \ \ x \to \infty$$

So, we need to find a bounded solution to this problem. So, let me call this equation as I and let me call these boundary and initial conditions as II. So, then let me straight away apply Laplace transform to I and II. Here I am applying Laplace transform with respect to the variable t and I see that I get the following expression:

$$\left[s^{\alpha}\bar{u}(s) - s^{\alpha-1}u(\alpha,0)\right] = \kappa\bar{u}_{xx}(s)$$

Before simplifying this further, I see that the solution at $t = 0$ is 0. So, first to my equation I, I see that my Laplace transform reduces to the following ODE:

$$\frac{d^2\bar{u}}{dx^2} = \frac{s^{\alpha}}{\kappa}\bar{u}(s); \qquad x > 0$$

Let me call this ODE as $I^{'}$. Similarly, I apply my Laplace transform to II again with respect to the variable t, I get the following equation:

$$\tilde{u}(0,s) = u_0\bar{F}(s)$$

So the solution must be bounded and so is the Laplace transform. Let me call this as $II^{'}$. So, solving $I^{'}$ and $II^{'}$, with subject to the condition that the solution must be bounded, I get the following solution to the ODE:

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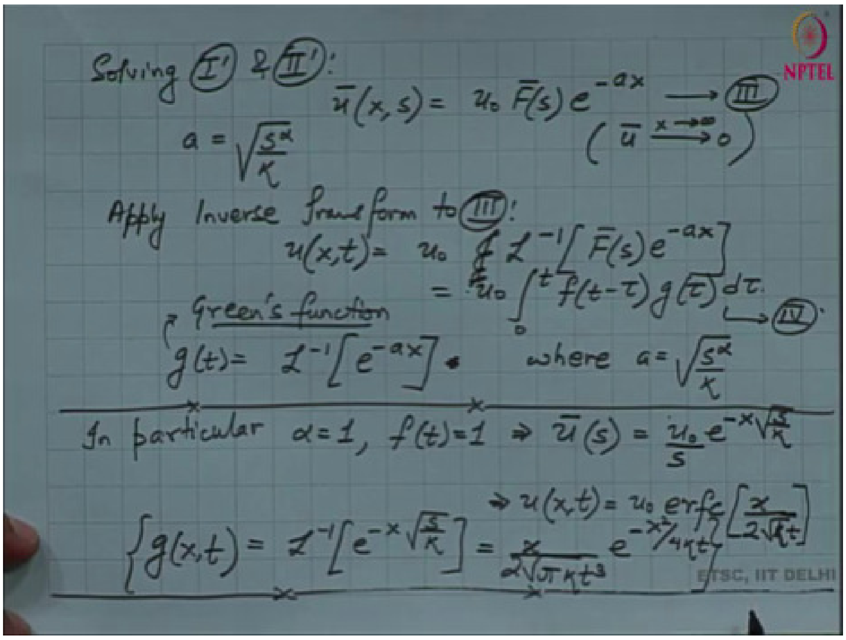

So, solving $I^{'}$ and $II^{'}$, I get the following solution to the ODE:

$$\tilde{u}(x,s) = u_0\bar{F}(s)e^{-ax}$$

where my constant $a = \sqrt{\frac{s^{\alpha}}{\kappa}}$. Notice that $\bar{u} \to 0$ as $x \to \infty$, so the solution is bounded. Thus only the negative exponents are retained.Let me call this as III. So, the next step is to apply inverse transform to III to come to the new solution in the physical plane as

$$u(x,t) = u_0\mathscr{L}^{-1}\left[\bar{F}(s)e^{-ax}\right]$$

So, that will be nothing but the inverse transform of the product of two Laplace transform and will give me convolution as follows:

$$= u_0 \int_0^t f(t-\tau) g(\tau) d\tau$$

where,

$$g(t) = \mathscr{L}^{-1}\left[e^{-ax}\right], \quad \text{where} \quad a = \sqrt{\frac{s^\alpha}{\kappa}}$$

So, here I have found the solution to the fractional diffusion equation which is given by IV. Now the further evaluation of this integral given by IV depends on the functional form of f as well as what is the order α. Now in particular, I can always discuss a particular case in particular. If I am given that $\alpha = 1$ and for simplification we choose $f = 1$ and we suppose that the functional form of both α and f are given, then:

$$\bar{u}(s) = \frac{u_0}{s} e^{-x\sqrt{s/\kappa}}$$

$$\text{Thus,} \qquad u(x,t) = u_0 \; erfc\left[\frac{x}{2\sqrt{\kappa t}}\right]$$

So, this is the solution here and further in this case I can write down my Green's function g, which is the Green's function in this semi infinite diffusion case. So, my Green's function is:

$$g(x,t) = \mathscr{L}^{-1}\left[e^{-x\sqrt{s/\kappa}}\right] = \frac{x}{2\sqrt{\pi \kappa t^3}} e^{-x^2/4\kappa t}$$

So, that completes the discussion here on fractional diffusion equation. Let me just move on to another example that is on fractional stokes/ fractional relay and fractional equations in fluid flows.

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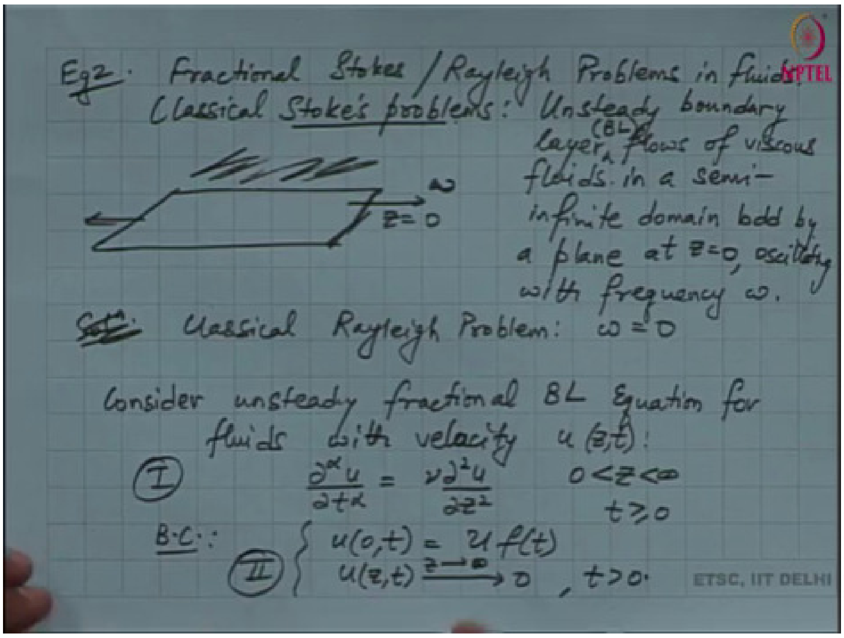

So, in the next example I am going to talk about fractional unsteady fractional Stoke?s and Rayleigh problem. So, let us look at some examples. Before I move ahead to look at the fractional case let me just highlight what is the classical case of Stoke?s and Rayleigh problems. So, the classical Stoke?s problem or the regular PDE which governs the Stoke?s phenomena is related to the unsteady boundary layers flows of viscous fluids in the semi infinite domain bounded by a plane and the plane is at $z = 0$ and oscillating with frequency ω.

So, we have some oscillation of the plate at this $z = 0$ and there is some fluid which is on top of this plate and the fluid we want to study is the behavior of this fluid using the standard Navier Stokes equation at a very low Reynold?s number or close to 0, so, zero Reynold?s number. So this is the classical Rayleigh problem which is when $\omega = 0$, that is when the plate is not moving at all and I want to study what is the behavior of the fluid above the plate.

So, let us now consider the fractional Stoke?s/Rayleigh problem. So, let us describe the behavior of the flow we have an unsteady fractional boundary layer equation for fluids with velocity given by $u(z,t)$. So, in that case my fractional equation is as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \nu \frac{\partial^2 u}{\partial z^2}, \qquad 0 < z < \infty, \quad t \geq 0$$

I am further given the boundary conditions as follows:

$$u(0,t) = uf(t)$$
$$u(z,t) \longrightarrow 0 \ \text{ as } \ z \to \infty, \quad t > 0$$

So, the fluid has a no slip boundary condition which means that the velocity of the fluid is equal to the velocity of the plate and the second condition means that the solution decays to 0 as we take the coordinate to infinity.

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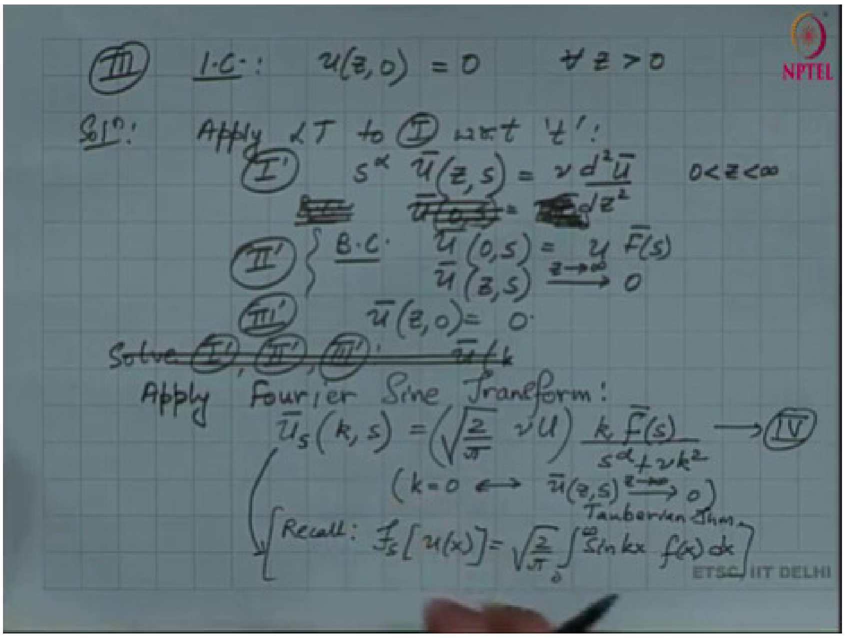

Let me call equation as I and let me call these boundary conditions as II and further we have the initial condition:

$$u(z,0) = 0, \quad \text{for all} \quad z > 0$$

Let me call this initial condition by III. Now, for the solution we apply my Laplace transform to I with respect to my variable t and when I do that I get the following ODE:

$$s^{\alpha}\bar{u}(z,s) = \nu\frac{d^2\bar{u}}{dz^2}, \quad 0 < z < \infty$$

and the boundary condition is:

$$\bar{u}(0,s) = u\bar{F}(s)$$
$$\bar{u}(z,s) \longrightarrow 0 \quad \text{as} \quad z \to \infty$$

So, this is my $I^{'}$ and my boundary conditions $II^{'}$ and of course, my initial condition is a $III^{'}$ and i.e.,

$$\bar{u}(z,0) = 0$$

Now, I see that this is an ODE, but I have the Laplace transform with respect to time and I have retained my physical variable. So, even before solving, let me transform even my physical spatial variable. Given the Dirichlet condition on the plate I am going to use my sine transform to transform this physical space z into the Fourier space. So, so, apply Fourier sine transform as follows to come to the complete transform domain:

$$\bar{u}_s(k,s) = \left(\sqrt{\frac{2}{\pi}}\nu u\right)\frac{k\bar{F}(s)}{s^{\alpha} + \nu k^2}$$

Here, s denotes the sine transformed solution of the transform variable (k,s) where, k is the transform variable with respect to space and s is the transform variable with respect to time. So, notice that this k is rising due to the fact that the solution is 0 at time point $t = 0$. So, which means that this particular form of this transformed solution takes care into account all my boundary conditions and initial conditions. So, I see that note that this $k = 0$ here is equivalent to saying that the solution $\bar{u}(z,s) = 0$ as $z \to \infty$. So, that is equivalent to that to our Taubarian theorem for the Fourier case.

So, then the next step is to apply the Fourier inverse transform. Before I move ahead I just want to highlight that this is the sine transform. So, let me just recap what is my sine transform:

$$f_s\big[u(x)\big] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin(kx)f(x)dx$$

This is particularly useful when we have a Dirichlet boundary condition which is the case in our situation. So, let me call this expression as IV and I want to invert this expression IV with respect to the Fourier and the Laplace transform.

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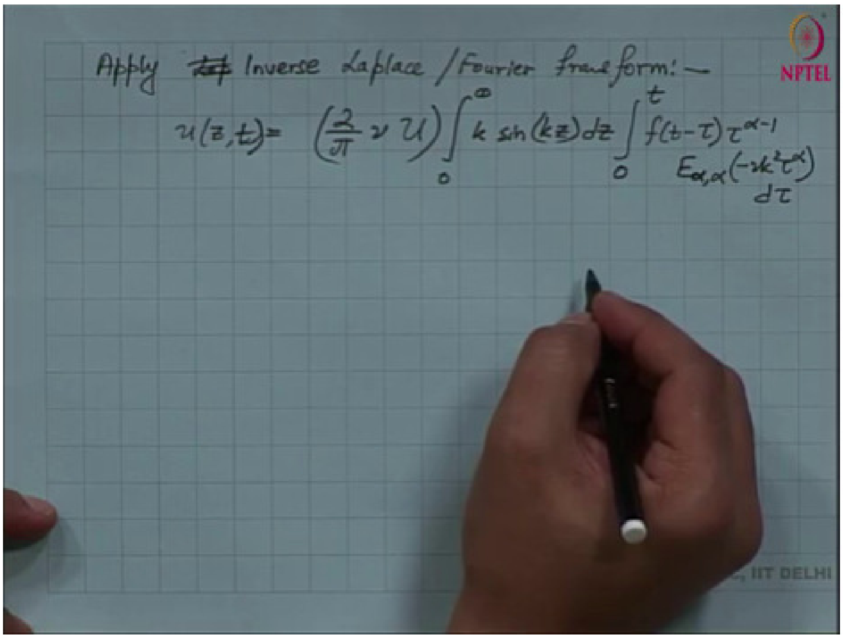

Let me just apply the inverse Laplace and Fourier transform. So, what I have is the following:

$$u(z,t) = \frac{2}{\pi}\nu u \int_0^\infty k\sin(kz)dz \int_0^t f(t-\tau)\tau^{\alpha-1}E_{\alpha,\alpha}(\nu k^2\tau^\alpha)d\tau$$

So, then we can proceed a little bit further by assuming certain functional forms of f. So, let me just look at some specific cases.