Integral Transforms and Their Applications

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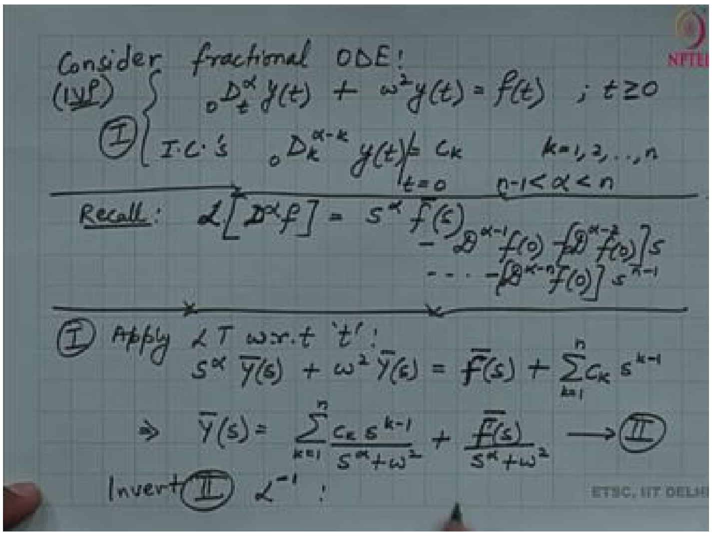
Lecture –61
Fractional PDEs Part 1

Good morning everyone. So, in the last lecture we saw how to solve certain ordinary differential equations the fractional ordinary differential equations based on namely three principles. The first one was the Lacroix formula for evaluating the derivatives, the fractional derivatives of certain polynomials, the second one was the Riemann-Liouville integral operator for fractional integrals and the third was how to evaluate integral and differential equations using the method of Laplace and inverse Laplace transform. So, in this lecture I am going to continue our discussion on solving certain fractional ODE's and also moving on in the latter half of the lecture towards fractional PDE's. So, let me start our discussion today.



So, again; so I am going to start by showing you an ODE of the general form. So let me consider a fractional ODE of the form

$$ {}_0D_t^{\alpha} y(t) + \omega^2 y(t) = f(t); t \geq 0 \qquad ...(\text{I}) $$

Initial Conditions are,

$$ {}_0D_k^{\alpha-k} y(t)|_{t=0} = c_k \quad k = 1, 2, ...n \quad ; n-1 < \alpha < n $$

Recall,

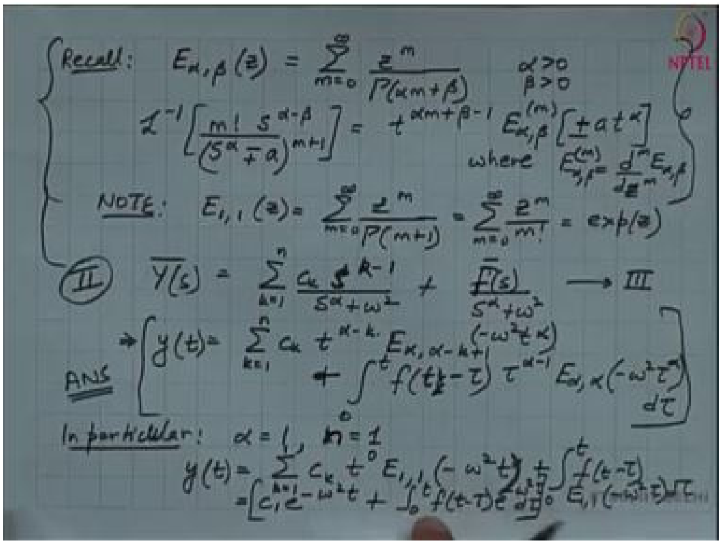
$$ \mathcal{L}\left[D^{\alpha} f\right] = s^{\alpha}\bar{f}(s) - D^{\alpha-1}f(0) - [D^{\alpha-2}f(0)]s ... - [D^{\alpha-n}f(0)]s^{n-1} $$

In (I) Apply Laplace Transform with respect to 't',

$$ s^{\alpha}\bar{Y}(s) + \omega^2\bar{Y}(s) = \bar{F}(s) + \sum_{k=1}^{n} c_k s^{k-1} $$

$$\Rightarrow \bar{Y}(s) = \sum_{k=1}^{n} \frac{c_k s^{k-1}}{s^\alpha + w^2} + \frac{\bar{F}(s)}{s^\alpha + w^2} \qquad ...(II)$$

Now take the invert of (II).we get,



Recall,

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)} \qquad \alpha > 0, \beta > 0$$

$$\mathcal{L}^{-1}\left[\frac{m! s^{\alpha-\beta}}{(s^\alpha \mp a)^{m+1}}\right] = t^{\alpha m+\beta-1} E^{(m)}_{\alpha,\beta}[\pm a t^\alpha] \qquad ....(II)$$

where,

$$E^{(m)}_{\alpha,\beta} = \frac{d^m}{dz^m} E_{\alpha,\beta}$$

$$\text{Note:} \quad E_{1,1}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m+1)} = \sum_{m=0}^{\infty} \frac{z^m}{m!} = exp(z)$$

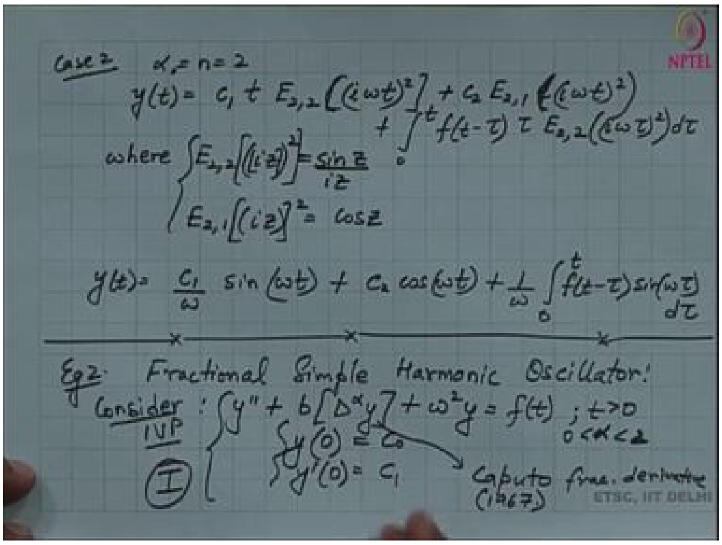
$$\bar{y}(s) = \sum_{k=1}^{n} \frac{a s^{k-1}}{s^\alpha + w^2} + \frac{\bar{F}(s)}{s^\alpha + w^2} \qquad ...(III)$$

$$y(t) = \sum_{k=1}^{n} c_k t^{\alpha-k} E_{k,\alpha-k+1}\left(-\omega^2 t^\alpha\right) + \int_0^t f\left(t-\tau\right) \tau^{\alpha-1} E_{\alpha,\alpha}\left(-\omega^2 \tau^\alpha\right) d\tau$$

In Particular, $\alpha = 1, n = 1$,

$$y(t) = \sum_{k=1}^{n} c_k t^0 E_{1,1}\left(-\omega^2 t\right) + \int_0^t f(t-\tau) E_1(-\omega^2 \tau) d\tau$$

$$= \left[c_1 e^{-w^2 t} + \int_0^t f(t-\tau) e^{-\omega^2 \tau} d\tau\right]$$



Case 2:$\alpha = n = 2$,

$$y(t) = c_1 t E_{2,2}\left[(i\omega t)^2\right] + c_2 E_{2,1}\left((i\omega t)^2\right) + \int_0^t f(t-\tau)\tau E_{2,2}(i\omega\tau)^2\Big)\, d\tau$$

Where,

$$E_{2,2}[(iz)]^2 = \tfrac{\sin z}{iz}$$
$$E_{2,1}[(iz)]^2 = \cos z$$

$$y(t) = \frac{c_1}{\omega}\sin(\omega t) + c_2\cos(\omega t) + \frac{1}{\omega}\int_0^t f(t-\tau)\sin(\omega\tau)\, d\tau$$

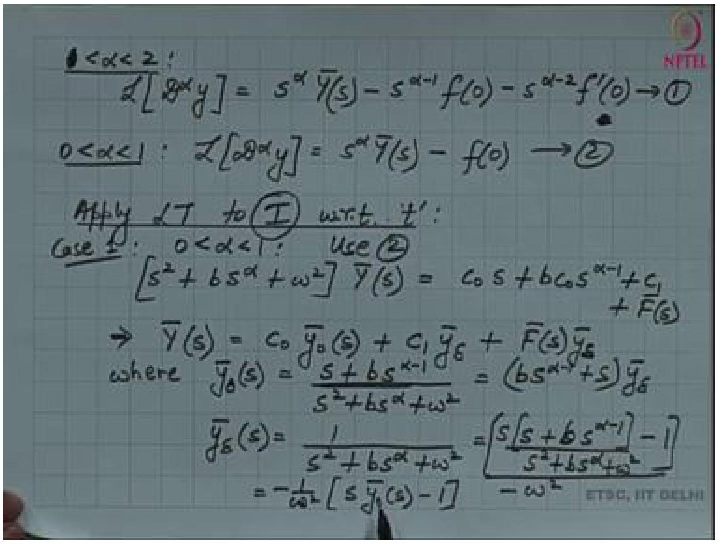
Example 2:Fractional Simple Harmonic oscillator,
Consider Initial Value Problem,

$$y'' + b\left[D^\alpha y\right] + \omega^2 y = f(t); t > 0 \quad 0 < \alpha < 2 \qquad ....(\text{I})$$

$$y(0) = c_0$$
$$y'(0) = c_1$$

So, these are my initial conditions. So, then before I move ahead I want to highlight the fact that we see that the difference between the regular simple harmonic oscillator and the fractional simple harmonic oscillator is this fractional derivative . So, this particular fractional derivative when it was solved the first time was solved by a person by the name of Caputo. So, Caputo his name is associated with this particular derivative in this SHM, it is also known as the Caputo's fractional derivative. So, this particular equation is Caputo's fractional derivative, first solved in 60's.

So, let us now look at and start to solve, let me call this as my (I) or I am going to start to solve this fractional ODE.



So, then before that let me just highlight the fact that
$\underline{0<\alpha<2}$:

$$\mathcal{L}[D^{\alpha}y] = s^{\alpha}\bar{y}(s) - s^{\alpha-1}f(0) - s^{\alpha-2}f'(0) \quad ....(1)$$

$\underline{0<\alpha<1}$:

$$\mathcal{L}\left[D^{\alpha}_{y}\right] = s^{\alpha}\bar{Y}(s) - f(0) \qquad ...(2)$$

Apply Laplace transform on (I) w.r.t 't',
Case 1:$\underline{0<\alpha<1}$: Use Equation(2)

$$\left[s^2 + bs^{\alpha} + w^2\right]\bar{Y}(s) = c_0 s + bc_0 s^{\alpha-1} + c_1 + F(s)$$

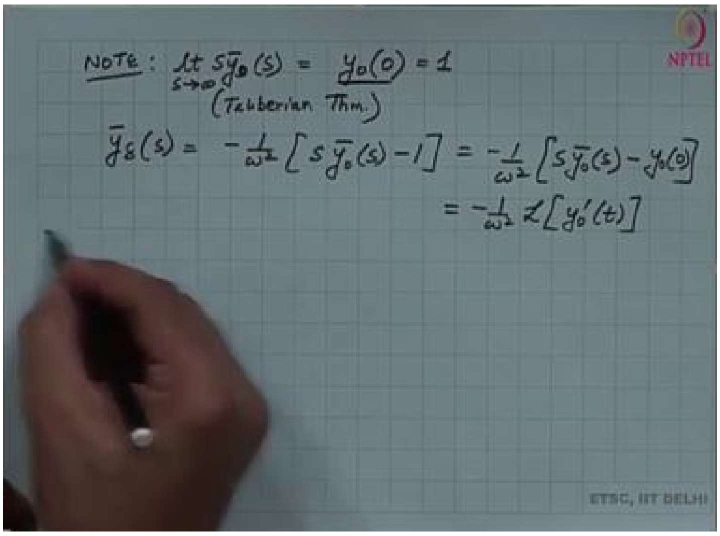
$$\Rightarrow \bar{Y}(s) = c_0\bar{y}_0(s) + c_1\bar{y}_{\delta} + \bar{F}(s)\bar{y}_{\delta}$$

Where,

$$y_{\delta}(s) = \frac{s + bs^{\alpha-1}}{s^2 + bs^{\alpha} + \omega^2} = \left(bs^{\alpha-1} + s\right)\bar{y}_{\delta}$$

$$\bar{y}_{\delta}(s) = \frac{1}{s^2 + bs^{\alpha} + \omega^2} = \frac{\left[\frac{s\left[s+bs^{\alpha-1}\right]}{s^2+bs^{\alpha}+\omega^2} - 1\right]}{-\omega^2}$$

$$= -\frac{1}{\omega^2}\left[s\bar{y}_0(s) - 1\right]$$



Note:So, I know from my Tauber's theorem which was done in my Laplace transform discussion, my Tauberian theorem gives me that

$$\lim_{s\to\infty} s\bar{y}_0(s) = y_0(0) = 1$$

$$\bar{y}_\delta(s) = -\frac{1}{\omega^2}\left[s\bar{y}_0(s) - 1\right]$$

$$= -\frac{1}{\omega^2}\left[s\overline{y_0}(s) - y_0(0)\right]$$

$$= -\frac{1}{\omega^2}\{[y_0'(t)]$$