Integral Transforms and Their Applications
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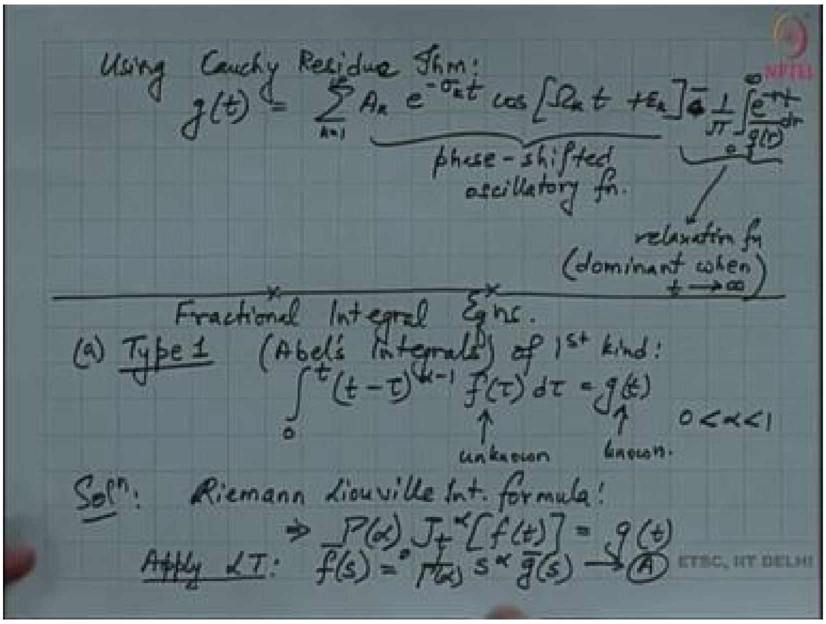
Lecture –59
Fractional ODEs, Abel's Integral Equations Part 2

Let me just move ahead and discuss about Fractional Integral Equations.so, fractional integral equations. So, I am going to start my discussion with my Abel's integral equations. So, there are mainly two different types of Abel's integral equation. So, let me start with the first type denoted by type 1. So, these are the Abel's integrals.



So, this is a first kind. So, what this integral is as follows:
a)Type 1: Abel's Integral of first kind,

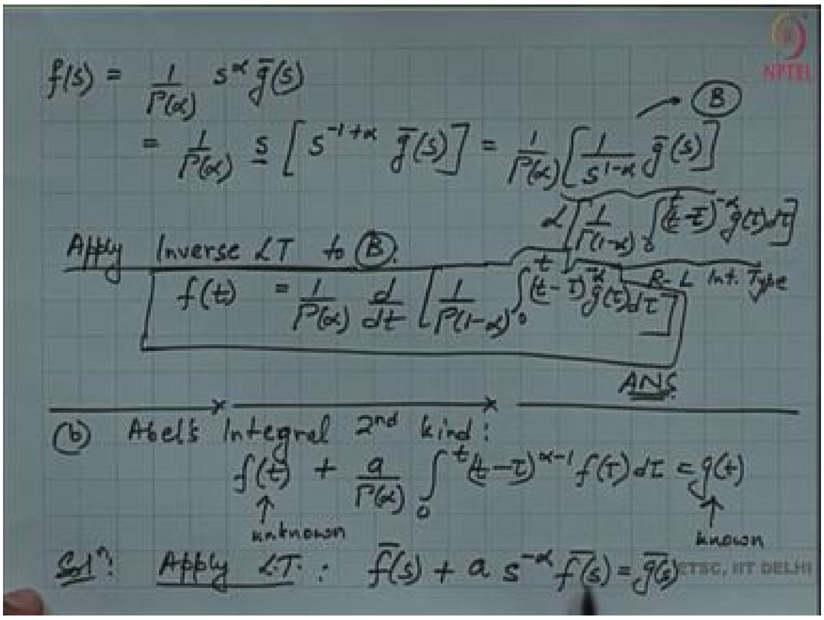
$$\int_0^t (t-\tau)^{k-1} f(\tau) d\tau = g(t) \qquad ; 0 < \alpha < 1$$

then if I were to solve this integral equation, I can right away use; I can right away use my Riemann-Liouville formula, Riemann-Liouville integral formula to come to the expression by apply; well we also apply the Laplace transform and then use the Riemann-Liouville integral formula. Solution:By Riemann Liouville's Integral Formulae,

$$\Rightarrow \Gamma(\alpha) J_t^{\alpha}[f(t)] = g(t)$$

Apply Laplace Transform,

$$\bar{f}(s) = \Gamma(\alpha) s^{\alpha} \bar{g}(s) \qquad ...(A)$$



$$f(s) = \frac{1}{\Gamma(\alpha)} s^{\alpha} \bar{g}(s)$$

$$= \frac{1}{\Gamma(\alpha)} s\left[s^{-1+\alpha}\bar{g}(s)\right] = \frac{1}{\Gamma(\alpha)}\left[\frac{1}{s^{1-\alpha}}\bar{g}(s)\right] \quad ..(B)$$

Apply Laplace Transform to (B),

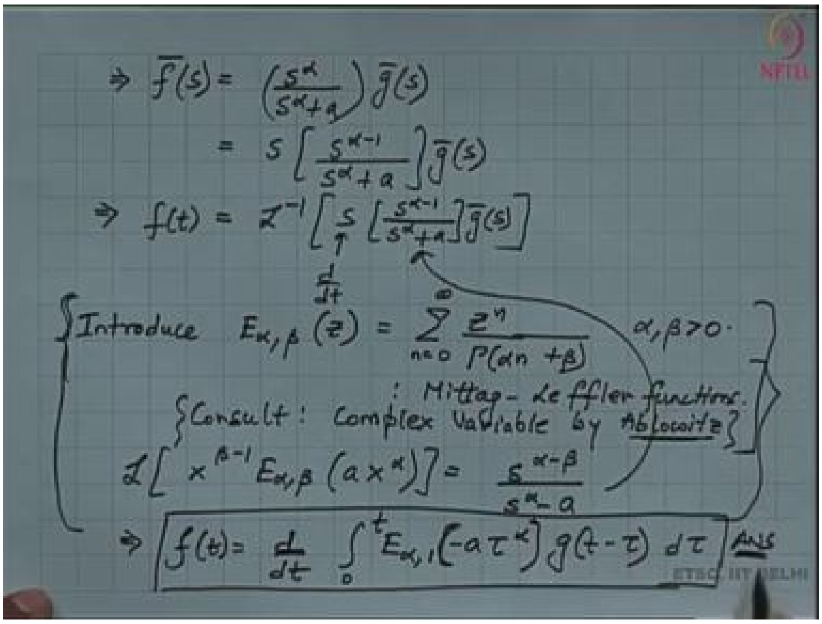
$$f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha} g(\tau)d\tau\right]$$

b) Abel's Integral 2nd Kind,

$$f(t) + \frac{a}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau = g(t)$$

Solution: Apply Laplace Transform,

$$\bar{f}(s) + a s^{-\alpha}\bar{f}(s) = \bar{g}(s)$$



$$\Rightarrow \bar{f}(s) = \left(\frac{s^\alpha}{s^\alpha + a}\right)\bar{g}(s)$$

$$= s\left[\frac{s^{\alpha-1}}{s^\alpha + a}\right]\bar{g}(s)$$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left[\frac{s^{\alpha-1}}{s^\alpha + a}\right]\bar{g}(s)\Big]$$

Now, let me before evaluating I can see the reason I have written it in this expression in this form is because the inverse transform will be the convolution well this s is going to bring in a derivative in the physical plane, but the derivative of the convolution of two functions one is this g and one is the inverse transform of this particular quantity.

So, let me just highlight what is this particular quantity, whose Laplace transform is in this small bracket. So, let me introduce another complex series as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad ; \alpha, \beta > 0$$

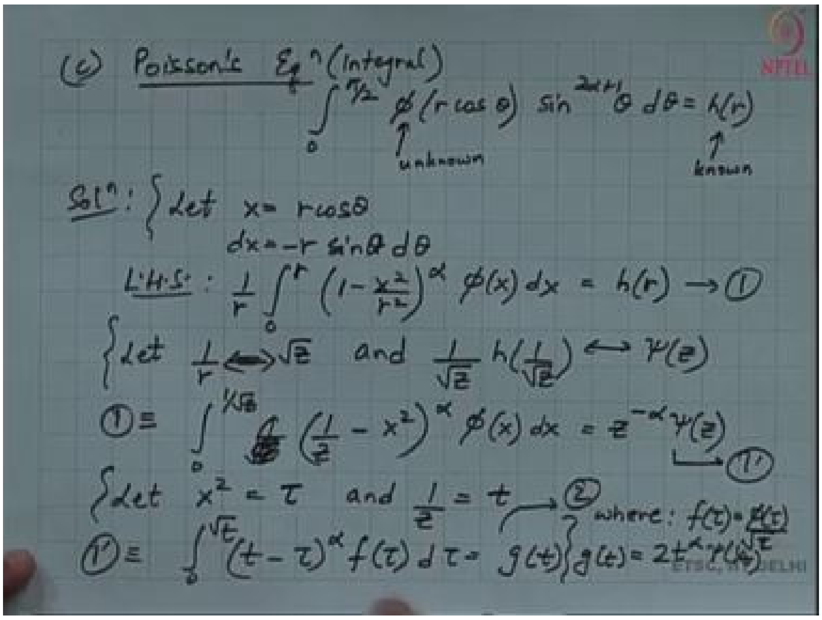
where my coefficients α and β are positive and what I have these are also known as the famous Mittag Leffler expansion.

So, these are the Mittag Leffler x functions ok. So, students are requested to look at the following text. So, please consult the following text complex variables by Abramowitz well by a Abramowitz ok. So, please consult this following text to come to this where these particular functions, these complex functions are nicely explained. So, I am going to use this function and I also want to highlight that the Laplace transform of these functions x to the power beta minus 1 of these Mittag Leffler functions are as follows.

$$\mathcal{L}\left[x^{\beta-1}E_{\alpha,\beta}\left(ax^\alpha\right)\right] = \frac{s^{\alpha-\beta}}{s^\alpha - a}$$

$$f(t) = \frac{d}{dt}\int_0^t E_{\alpha,1}\left(-a\tau^{\alpha}\right) g(t-\tau)d\tau$$

So, that is the solution to this problem the Abel's integral equation of the second type.



c)let me just highlight one specific case of Abel's integral equation namely the Poisson equation, the Poisons equation. So, Poisons equation says that I have the following integral. So, these this is a Poisson integral equation not the regular Poisson equation that we are aware of. So, this integral equation is as follows:

$$\int_0^{\pi/2} \phi(r\cos\theta)\sin^{2\alpha+1}\theta d\theta = h(r)$$

Solution:Let $x = r\cos\theta$, $dx = -r\sin\theta d\theta$

$$LHS : \frac{1}{r}\int_0^r \left(1-\frac{x^2}{r^2}\right)^{\alpha}\phi(x)dx = h(r) \qquad ...(1)$$

$$\text{Let } \frac{1}{r} \Leftrightarrow \sqrt{z} \text{ and } \frac{1}{\sqrt{z}}h\left(\frac{1}{\sqrt{z}}\right) \leftrightarrow \psi(z)$$

Equation (1) can be written as,

$$(1) = \int_0^{1/\sqrt{z}} \left(\frac{1}{z}-x^2\right)^{\alpha}\phi(x)dx = z^{-\alpha}\psi(z) \qquad ....(1')$$

$$\text{Let } x^2 = \tau \quad \text{ and } \frac{1}{z} = t$$

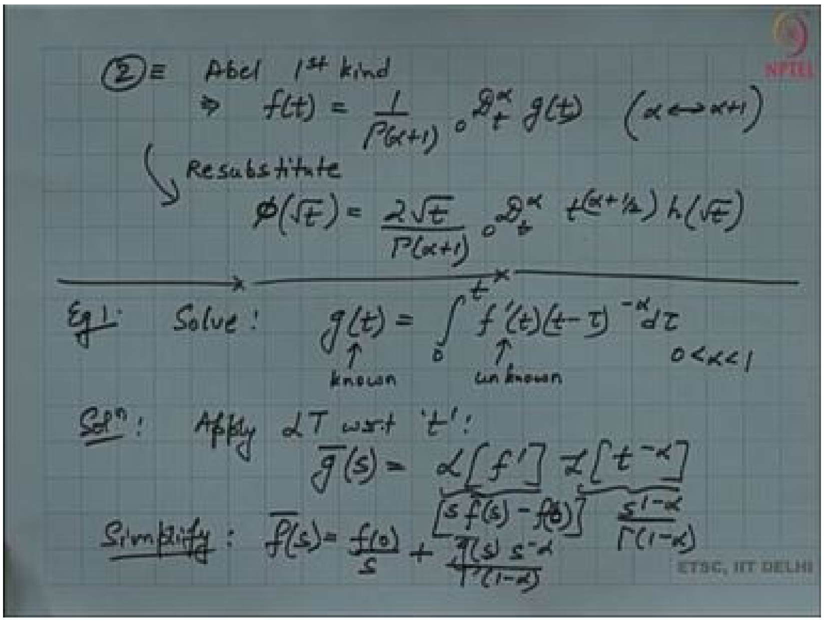
Now, (1') can be transformed as,

$$(1\text{'}) = \int_0^{\sqrt{t}} (t-\tau)^{\alpha} f(\tau)d\tau = g(t) \qquad ..(2)$$

$$\text{where,} f(\tau) = \frac{\phi(\tau)}{\sqrt{\tau}}$$

$$\text{Also,} g(t) = 2t^{\alpha}\psi(\phi(t))$$

I see that 2 is nothing, but the Abel's integral of the first kind.



So, I can immediately recognize that two is equivalent to Abel's first kind of integral that we have solved and I can immediately find the answer to my equation that is

$$f(t) = \frac{1}{\Gamma(\alpha+1)} {}_0D_t^{\alpha} g(t) \quad ; (\alpha \Leftrightarrow \alpha+1)$$

After Re-substitute,So,as far we have made three sets of substitutions, then we have to go back and substitute back our original variables to come back to my expression for the unknown the unknown was $\phi(t)$. So, I get:

$$\phi(\sqrt{t}) = \frac{2\sqrt{t}}{\Gamma(\alpha+1)} {}_0D_t^{\alpha} + (t^{\alpha+1/2})h(\sqrt{t})$$

Example 1:Solve,

$$g(t) = \int_0^t f'(t)(t-\tau)^{-\alpha} d\tau \qquad ; 0 < \alpha < 1$$

Solution:Apply Laplace Transform w.r.t 't',

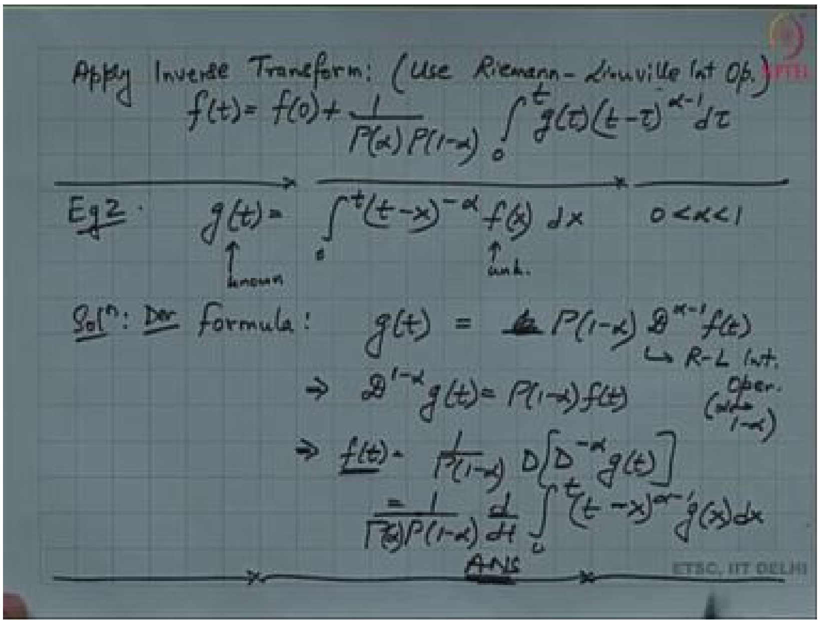
$$\bar{g}(s) = \mathcal{L}\left[f'\right]\mathcal{L}\left[t^{-\alpha}\right]$$

$$= [s\bar{f}(s) - f(0)]\frac{s^{1-\alpha}}{\Gamma(1-\alpha)}$$

Simplify,

$$\overline{f}(s) = \frac{f(0)}{s} + \frac{\bar{g}(s)s^{-\alpha}}{\Gamma(1-\alpha)}$$

this is corresponding to my fractional integral of the Riemann-Liouville type.



So, let me just apply the inverse transform to get back my result,

$$f(t) = f(0) + \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t g(\tau)(t-\tau)^{\alpha-1} d\tau$$

Example 2:

$$g(t) = \int_0^t (t-x)^{-\alpha} f(x) dx \quad ; 0 < \alpha < 1$$

Solution:

$$g(t) = \Gamma(1-\alpha) D^{\alpha-1} f(t)$$

I have also used my Riemann, my Riemann-Liouville integral operator formula.

$$D^{1-\alpha} g(t) = \Gamma(1-\alpha) f(t)$$

$$\Rightarrow f(t) \frac{1}{\Gamma(1-\alpha)} D \left[D^{-\alpha} g(t) \right]$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \left[(t-x)^{\alpha-1} g(x) dx \right]$$

So, that is the answer for the unknown f(t). So, then moving on let us look at some other examples.