**Integral Transforms and Their Applications**
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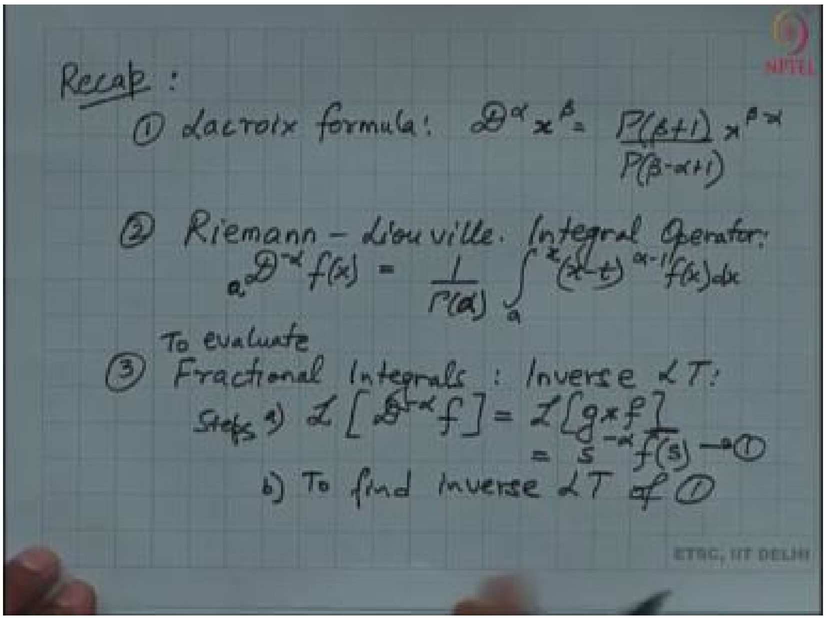
**Lecture –58**

# Fractional ODEs, Abel's Integral Equations Part 1




Recap:

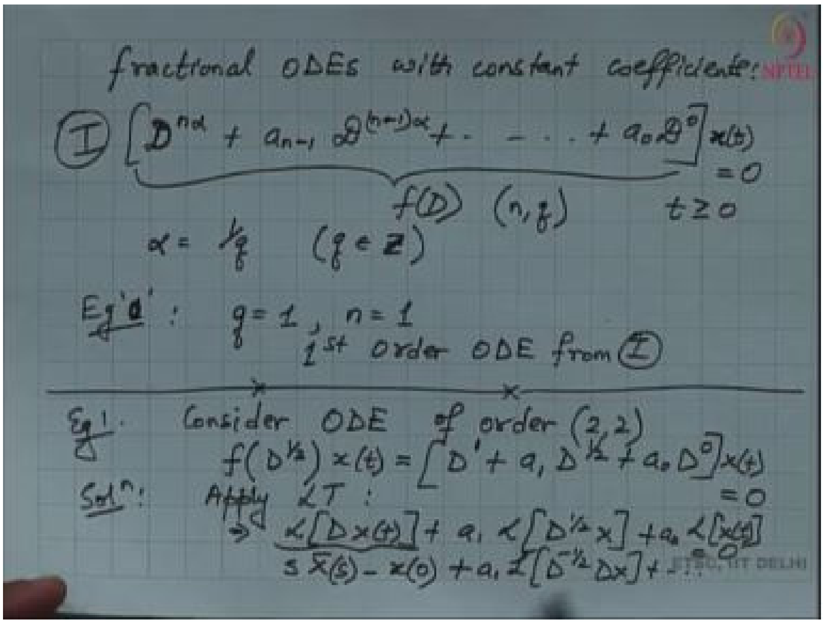
$$(1)\ \text{Lacroix formula: } D^{\alpha}x^{\beta} = \frac{P(\beta+1)}{P(\beta-\alpha+1)}x^{\beta\times}$$

Riemann Liouville Integral:

$$D^{-\alpha}f(x) = \frac{1}{r(a)}\int_a^x (x-t)^{\alpha-1}f(x)dx$$

To evaluate:

a) $L[D^{-\alpha}f] = L[g \times f]s^{-\alpha}f(s)$
b) To find inverse $\alpha T$ of 1




$$
\left. D^{n\alpha} + a_{n-1} D^{(n-1)\alpha} + \cdots + a_0 D^0 \right] \begin{array}{l} x(t) \\ = 0 \end{array}
$$

Example 0:

$$
g = 1, \quad n = 1\text{st order ODE from I}
$$

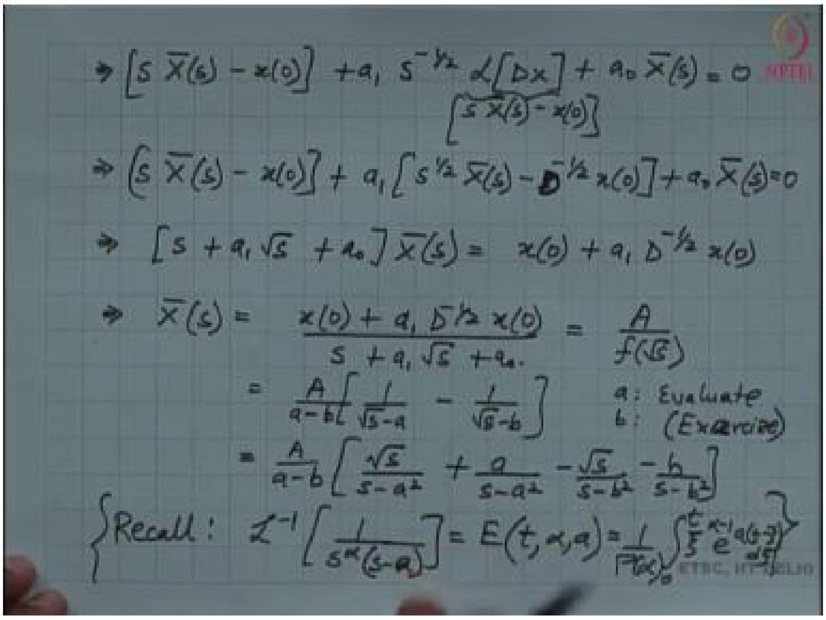
Example 1: Consider ODE of order(2,2):

$$
f\left(D^{\frac{1}{2}}\right) x(t) = \left[D' + a_1 D^{\left(\frac{1}{2}\right)} + a_0 \Delta^0\right] x(t) = 0
$$

Solution: Apply LT:

$$
L[Dx(t)] + a_1 L\left[D^{1/2} x\right] + a_0 L\left[x(t)\right] = 0
$$

$$
s\bar{x}(s) - x(0) + a_1 L\left[D^{-1/2} Dx\right] +
$$



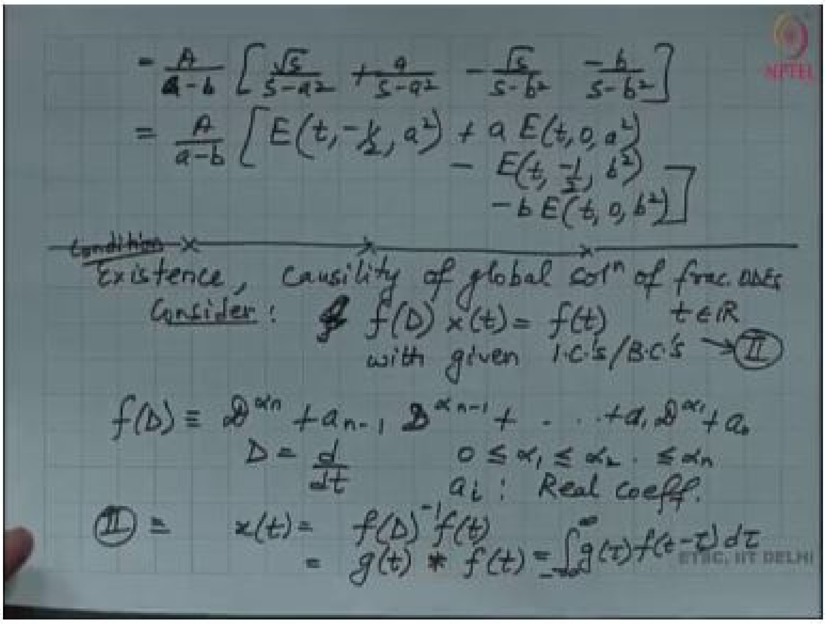
$$\Rightarrow [s\bar{x}(s) - x(0)] + a_1 s^{-1/2} L[Dx] + a_0 \bar{x}(s) = 0$$

$$\Rightarrow (s\bar{x}(s) - x(s)] + a_1 \left[s^{1/2} x(s) - D^{-1/2} x(0)\right] + a_1 \bar{x}(s) = 0$$

$$\Rightarrow \left[s + a_1\sqrt{s} + a_0\right] \bar{x}(s) = x(0) + a_1 D^{-1/2} x(0)$$

$$\Rightarrow \bar{x}(s) = \frac{x(0) + a_1 D^{-1/2} x(0)}{s + a_1\sqrt{s} + a_0} = \frac{A}{f(\sqrt{s})}$$

$$= \frac{A}{a-b}\left[\frac{1}{\sqrt{s-a}} - \frac{1}{\sqrt{s-b}}\right]$$




$$= \frac{A}{a-b}\left[\frac{\sqrt{s}}{s-a^2} + \frac{a}{s-a^2} - \frac{\sqrt{s}}{s-b^2} - \frac{b}{s-b^2}\right]$$

$$= \frac{A}{a-b}\left[E\left(t, -\frac{1}{2}, a^2\right) + aE\left(t, 0, a^2\right) - E\left(t, -\frac{1}{2}, b^2\right) - bE(t, 0, b^2)\right]$$

Consider:

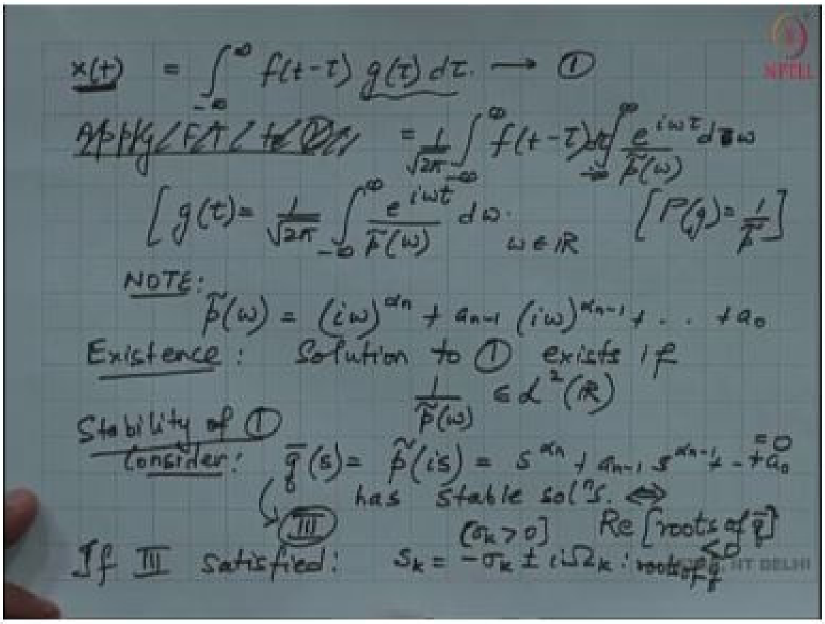
$$f(D) \times (t) = f(t) \quad t \in \mathbb{R}$$

with given $1.CS/B.C'.7(\mathbb{III})$

$$f(D) \equiv \mathscr{D}^{\alpha_n} + a_{n-1} D^{\alpha_{n-1}} + \ldots + a_1 \cdot D^{\alpha_1} + a_0$$

$$\mathbb{III} = x(t) = f(D)^{-1} f(t)$$

$$= g(t) * f(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau$$




$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-\tau)d\tau \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\tilde{p}(\omega)}d\omega$$

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{iwt}}{\tilde{p}(w)}d\omega$$

NOTE:

$$\tilde{p}(\omega) = (i\omega)^{\alpha_n} + a_{n-1}(i\omega)^{\alpha_{n-1}} + \cdots + a_0$$

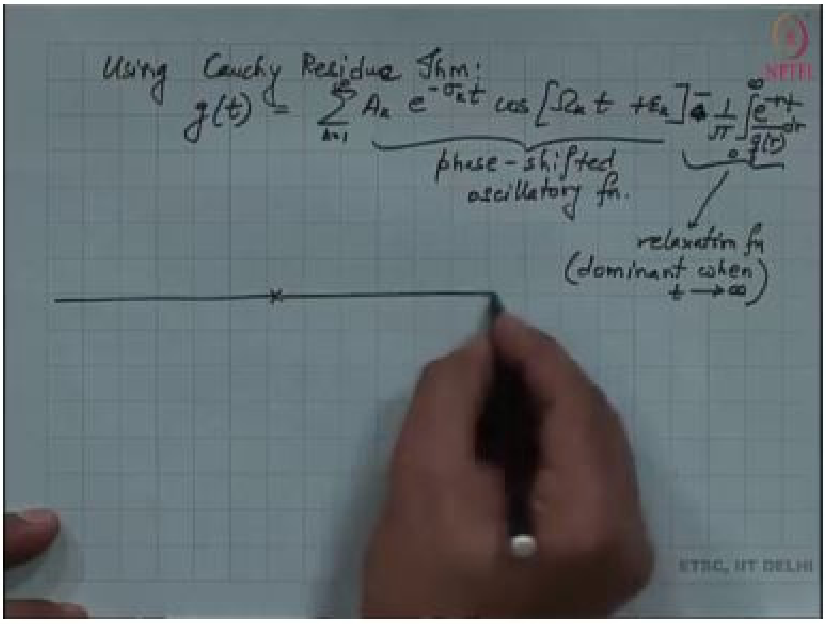
$$\frac{\text{Solution to 1 exists if}}{\frac{1}{\tilde{P}(x)} \in \alpha^2(R)} \text{ if}$$

Consider :

$$\bar{q}(s) = \tilde{p}(is) = s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \cdots + a_0 = 0$$

has stable solution. If Satisfied:

$$s_k = -\alpha_k + i\omega k\text{:roots of q}$$




$$g(t) = \sum_{k=1}^{n} A_x e^{-\sigma_k t} \cos[\omega_k t + \varepsilon_k] - \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-rt}$$