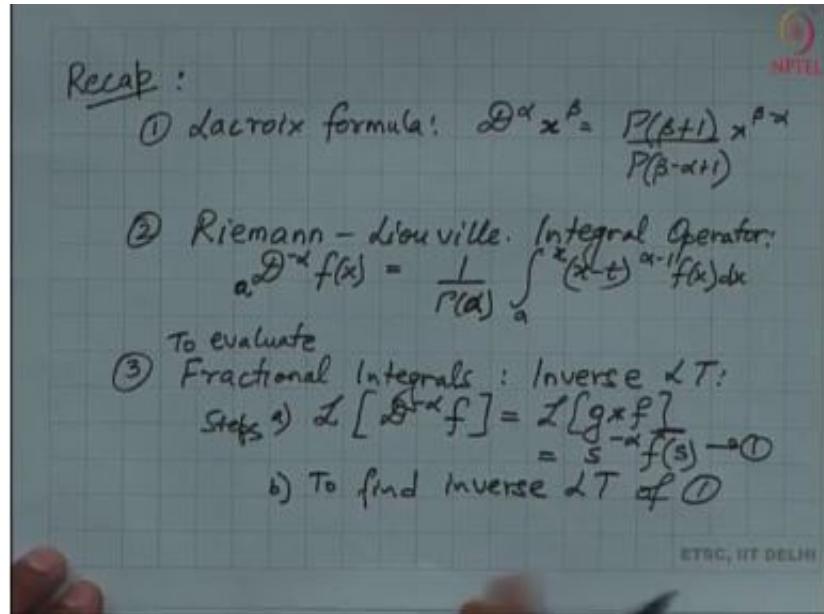


Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture -58

Fractional ODEs, Abel's Integral Equations Part 1



Recap:

$$(1) \text{ Lacroix formula: } D^\alpha x^\beta = \frac{P(\beta + 1)}{P(\beta - \alpha + 1)} x^{\beta + 1}$$

Riemann Liouville Integral:

$$D^{-\alpha} f(x) = \frac{1}{r(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

To evaluate:

- a) $L[D^{-\alpha} f] = L[g \times f] s^{-\alpha} f(s)$
- b) To find inverse αT of 1

fractional ODEs with constant coefficients! NPTEL

I $\left[D^{n\alpha} + a_{n-1} D^{(n-1)\alpha} + \dots + a_0 D^0 \right] x(t) = 0$

$\alpha = \frac{1}{g}$ ($g \in \mathbb{Z}$)

Eg: $g=1, n=1$
1st order ODE from I

Eg 1. Consider ODE of order (2,2)

Sol: Apply LT:

$$f(D^{\frac{1}{2}}) x(t) = [D' + a_1 D^{\frac{1}{2}} + a_0 \Delta^0] x(t) = 0$$

$$\Rightarrow [s \bar{x}(s) - x(0)] + a_1 [s^{\frac{1}{2}} \bar{x}(s)] + a_0 \bar{x}(s) = 0$$

IIT DELHI

$$D^{n\alpha} + a_{n-1} D^{(n-1)\alpha} + \dots + a_0 D^0 \Big] x(t) = 0$$

Example 0:

$g = 1, n = 1$ st order ODE from I

Example 1: Consider ODE of order(2,2):

$$f\left(D^{\frac{1}{2}}\right) x(t) = \left[D' + a_1 D^{\left(\frac{1}{2}\right)} + a_0 \Delta^0 \right] x(t) = 0$$

Solution: Apply LT:

$$L[Dx(t)] + a_1 L[D^{1/2}x] + a_0 L[x(t)] = 0$$

$$s\bar{x}(s) - x(0) + a_1 L[D^{-1/2}Dx] +$$

$$\Rightarrow [s \bar{x}(s) - x(0)] + a_1 s^{-\frac{1}{2}} L[Dx] + a_0 \bar{x}(s) = 0$$

$\boxed{s \bar{x}(s) - x(0)}$

$$\Rightarrow [s \bar{x}(s) - x(0)] + a_1 [s^{\frac{1}{2}} \bar{x}(s) - D^{-\frac{1}{2}} x(0)] + a_0 \bar{x}(s) = 0$$

$$\Rightarrow [s + a_1 \sqrt{s} + a_0] \bar{x}(s) = x(0) + a_1 D^{-\frac{1}{2}} x(0)$$

$$\Rightarrow \bar{x}(s) = \frac{x(0) + a_1 D^{-\frac{1}{2}} x(0)}{s + a_1 \sqrt{s} + a_0} = \frac{A}{f(s)}$$

$$= \frac{A}{a-b} \left[\frac{1}{\sqrt{s-a}} - \frac{1}{\sqrt{s-b}} \right]$$

$a: \text{Evaluate}$
 $b: (\text{Exercise})$

$$= \frac{A}{a-b} \left[\frac{\sqrt{s}}{s-a^2} + \frac{a}{s-a^2} - \frac{\sqrt{s}}{s-b^2} - \frac{b}{s-b^2} \right]$$

Recall: $\mathcal{L}^{-1} \left[\frac{1}{s^{\alpha} (s-a)} \right] = E(t, \alpha, a) = \frac{1}{\Gamma(\alpha)} \int_0^t e^{a(t-s)} \frac{s^{\alpha-1}}{s^{\alpha}} ds$

$$\begin{aligned}
& \Rightarrow [s\bar{x}(s) - x(0)] + a_1 s^{-1/2} L[Dx] + a_0 \bar{x}(s) = 0 \\
& \Rightarrow (s\bar{x}(s) - x(s)) + a_1 [s^{1/2}x(s) - D^{-1/2}x(0)] + a_0 \bar{x}(s) = 0 \\
& \Rightarrow [s + a_1 \sqrt{s} + a_0] \bar{x}(s) = x(0) + a_1 D^{-1/2}x(0) \\
& \Rightarrow \bar{x}(s) = \frac{x(0) + a_1 D^{-1/2}x(0)}{s + a_1 \sqrt{s} + a_0} = \frac{A}{f(\sqrt{s})} \\
& = \frac{A}{a-b} \left[\frac{1}{\sqrt{s-a}} - \frac{1}{\sqrt{s-b}} \right]
\end{aligned}$$

NPTEL

$$\begin{aligned}
& = \frac{A}{a-b} \left[\frac{\sqrt{s}}{s-a^2} + \frac{a}{s-a^2} - \frac{\sqrt{s}}{s-b^2} - \frac{b}{s-b^2} \right] \\
& = \frac{A}{a-b} \left[E(t, -\frac{1}{2}, a^2) + a E(t, 0, a^2) - E(t, -\frac{1}{2}, b^2) - b E(t, 0, b^2) \right]
\end{aligned}$$

Condition for existence, causality of global soln of frac. ODEs

Consider: $f(D) \times (t) = f(t) \quad t \in \mathbb{R}$
with given I.C's/B.C's $\Rightarrow \text{II}$

$f(D) \equiv D^{\alpha_n} + a_{n-1} D^{\alpha_{n-1}} + \dots + a_1 D^{\alpha_1} + a_0$
 $D = \frac{d}{dt} \quad 0 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_n$
 $a_i : \text{Real coeff.}$

$\text{II} \equiv x(t) = f(D)^{-1} f(t)$
 $= g(t) * f(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau$

$$\begin{aligned}
& = \frac{A}{a-b} \left[\frac{\sqrt{s}}{s-a^2} + \frac{a}{s-a^2} - \frac{\sqrt{s}}{s-b^2} - \frac{b}{s-b^2} \right] \\
& = \frac{A}{a-b} \left[E \left(t, -\frac{1}{2}, a^2 \right) + a E \left(t, 0, a^2 \right) - E \left(t, -\frac{1}{2}, b^2 \right) - b E \left(t, 0, b^2 \right) \right]
\end{aligned}$$

Consider:

$$f(D) \times (t) = f(t) \quad t \in \mathbb{R}$$

with given I.CS/B.C'S. 7(III)

$$f(D) \equiv D^{\alpha_n} + a_{n-1} D^{\alpha_{n-1}} + \dots + a_1 \cdot D^{\alpha_1} + a_0$$

$$\begin{aligned}
\text{III} &= x(t) = f(D)^{-1} f(t) \\
&= g(t) * f(t) = \int_{-\infty}^{\infty} g(\tau) f(t-\tau) d\tau
\end{aligned}$$

MITEE

$$\underline{x}(t) = \int_{-\infty}^{\infty} f(t-\tau) g(\tau) d\tau \rightarrow \textcircled{1}$$

$$\text{If } \tilde{p}(s) \neq 0 \text{ then } \textcircled{2} \text{ is } \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-\tau) d\tau \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\tilde{p}(\omega)} d\omega$$

$$[g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\tilde{p}(\omega)} d\omega, \omega \in \mathbb{R}, P(g) = \frac{1}{\tilde{p}(s)}]$$

NOTE: $\tilde{p}(\omega) = (i\omega)^{\alpha_n} + a_{n-1}(i\omega)^{\alpha_{n-1}} + \dots + a_0$

Existence: Solution to \textcircled{1} exists if $\frac{1}{\tilde{p}(\omega)} \in L^2(\mathbb{R})$

Stability of \textcircled{1}: Consider: $\bar{q}(s) = \tilde{p}(is) = s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \dots + a_0$

(\textcircled{3}) has stable sol's. \Leftrightarrow $a_n > 0$, Re[roots of \bar{q}] < 0

If \textcircled{3} satisfied: $s_k = -\sigma_k \pm i\omega_k$: roots of \bar{q}

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t-\tau) d\tau \int_{-\infty}^{\infty} \frac{e^{i\omega\tau}}{\tilde{p}(\omega)} d\omega$$

$$g(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\tilde{p}(\omega)} d\omega$$

NOTE:

$$\tilde{p}(\omega) = (i\omega)^{\alpha_n} + a_{n-1}(i\omega)^{\alpha_{n-1}} + \dots + a_0$$

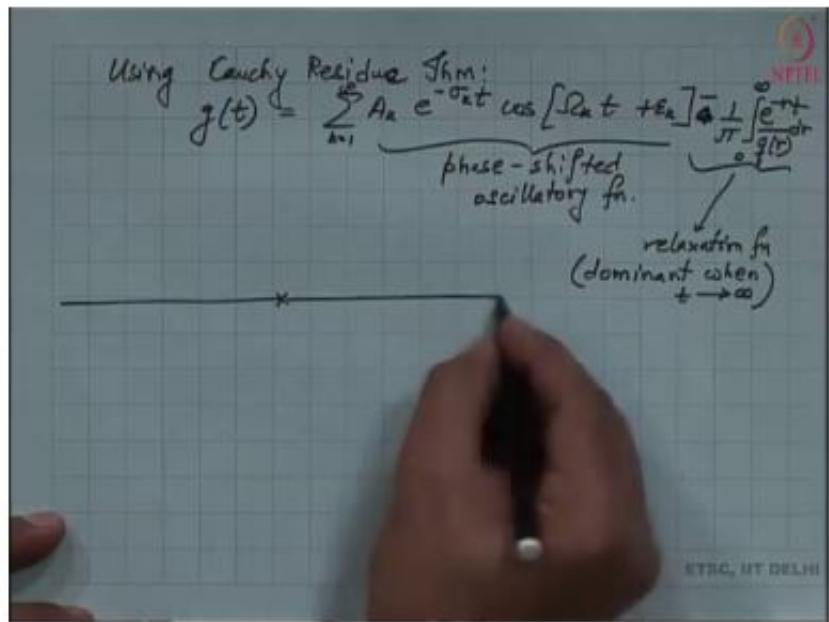
Solution to \textcircled{1} exists if
 $\frac{1}{\tilde{p}(s)} \in \mathcal{A}^2(R)$ if

Consider :

$$\bar{q}(s) = \tilde{p}(is) = s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \dots + a_0 = 0$$

has stable solution. If Satisfied:

$$s_k = -\alpha_k + i\omega_k : \text{roots of } q$$



$$g(t) = \sum_{k=1}^n A_k e^{-\sigma_k t} \cos[\omega_k t + \varepsilon_k] - \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-rt}$$