Integral Transforms and Their Applications

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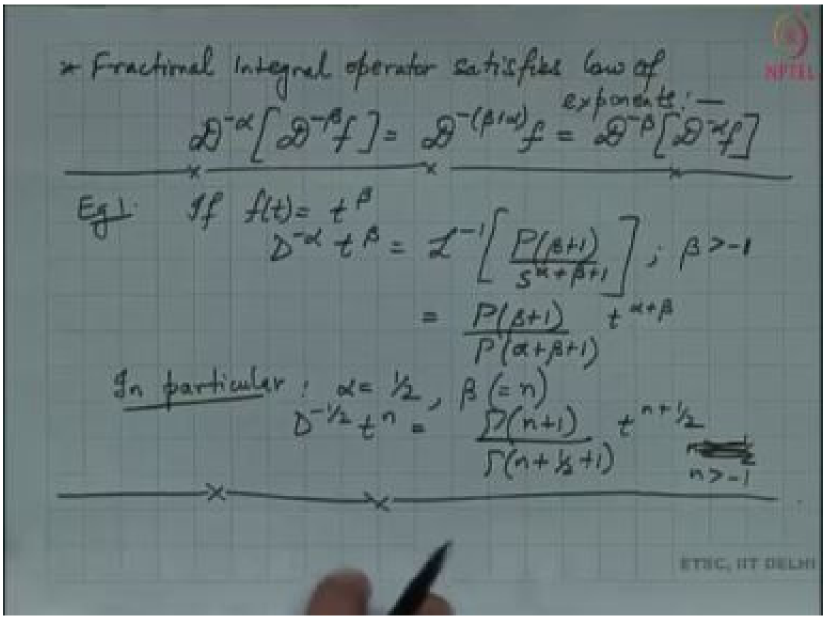
Lecture –57

Introduction to Fractional Calculus Part 3



$$D^{-\alpha}\left[D^{-\beta} f\right] = D^{-(\beta+\alpha)} f = D^{-\beta}[D^{-\alpha} f]$$

Example 1:

$$\text{If } f(t) = t^{\beta}$$

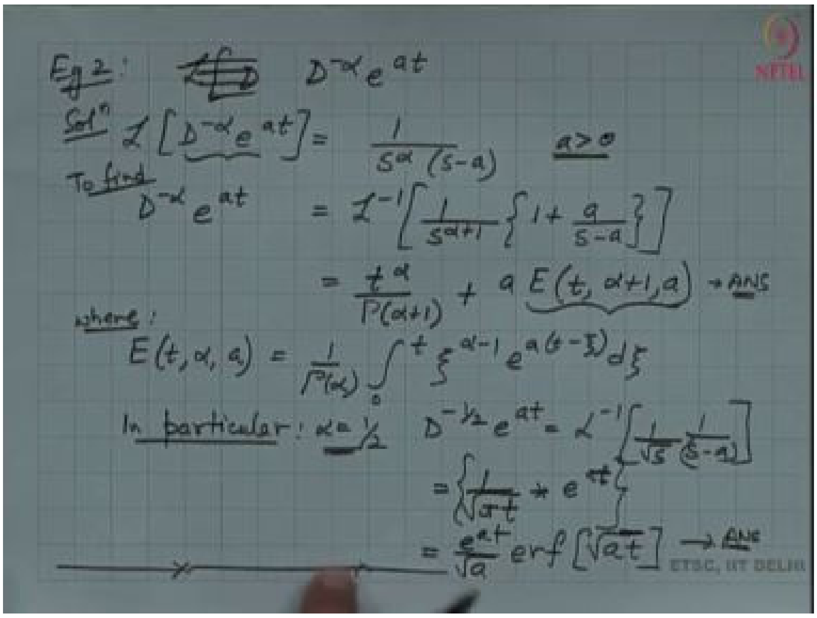
$$D^{-\alpha} t^{\beta} = \mathcal{L}^{-1}\left[\frac{P(\beta+1)}{S^{\alpha+\beta+1}}\right]; \beta > -1$$

$$= \frac{P(\beta+1)}{P(\alpha+\beta+1)} t^{\alpha+\beta}$$

In particular:

$$\alpha = 1/2, \beta(= n)$$

$$D^{-1/2} t^{n} = \frac{P(n+1)}{P(n+\frac{1}{2}+1)} t^{n+1/2}$$



Example 2:

$$D^{-\alpha}e^{at}$$

Solution:

$$L[D^{-\alpha}e^{at}] = \frac{1}{S^{\alpha}(s-a)} \; {\scriptstyle a>0}$$

$$D^{-\alpha}e^{\alpha t} = L^{-1}\left[\frac{1}{s^{\alpha+1}}\left\{1+\frac{a}{s-a}\right\}\right]$$

$$= \frac{t^{\alpha}}{P(\alpha+1)} + aE(t,\alpha+1,a)$$

where:

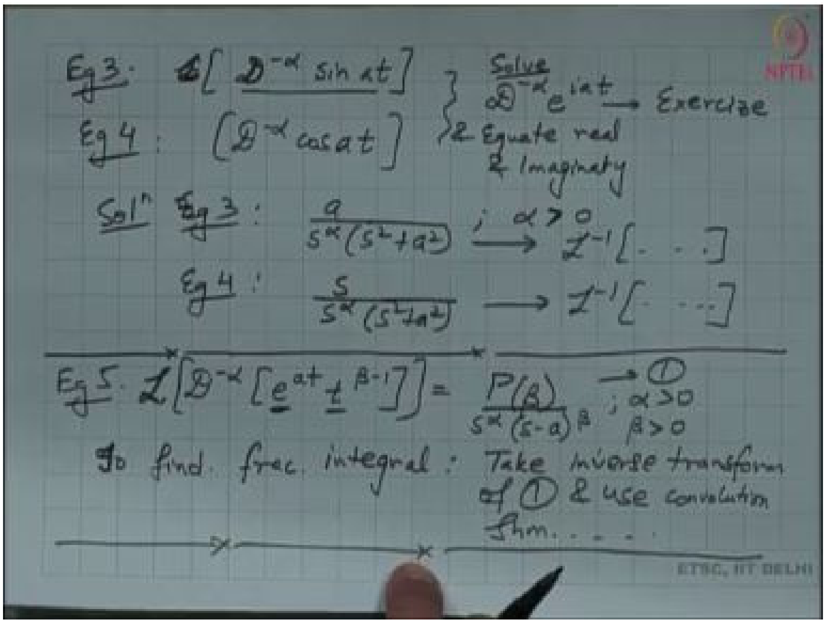
$$E(t,\alpha,a) = \frac{1}{P(\alpha)}\int_0^t \xi^{\alpha-1}e^{a(t-\xi)}d\xi$$

In particular:

$$\alpha = 1/2 \quad D^{-1/2}e^{at} = L^{-1}\left[\tfrac{1}{\sqrt{s}}\tfrac{1}{(s-a)}\right]$$

$$= \left\{\tfrac{1}{\sqrt{at}} + e^{at}\right\}$$

$$= \tfrac{e^{at}}{\sqrt{a}}\ \text{erf}\ [\sqrt{at}]$$



Example 3:

$$\left\{D^{-\alpha}\sin \alpha t\right\}$$

Example 4:

$$\left[D^{-\alpha}\cos at\right]$$

Solution: Example 3:

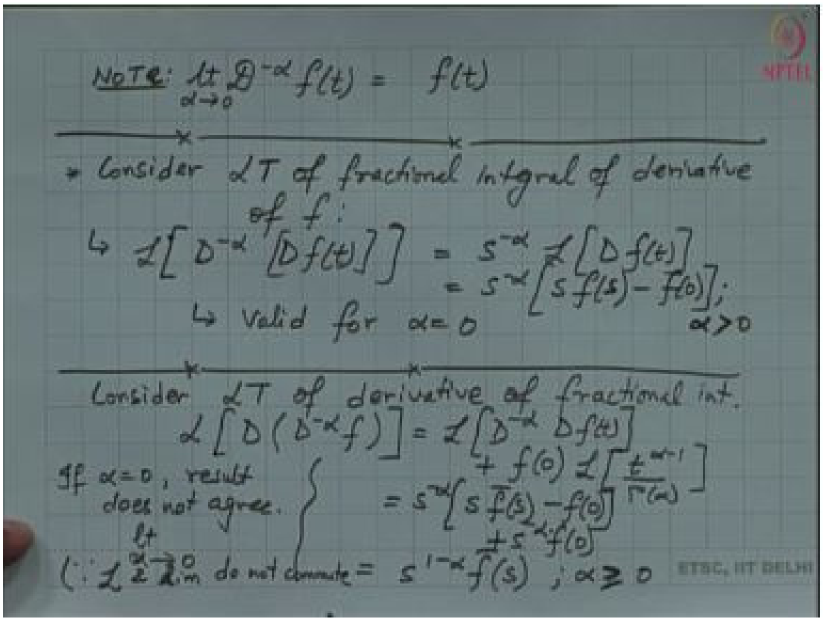
$$\frac{a}{s^{\alpha}\left(s^2+a^2\right)} \stackrel{\alpha>0}{\to} L^{-1}\left[\ldots\right]$$

Example 4:

$$\frac{s}{s^{2}\left(s^2+a^2\right)} \to L^{-1}[\cdots]$$

Example 5:

$$L\left[D^{-\alpha}\left[e^{at}t^{\beta-1}\right]\right]=\frac{P(\beta)}{s^{\alpha}(s-a)^{\beta}} \to_{\beta>0}^{\alpha>0}$$



$$\lim_{\alpha \to 0} D^{-\alpha} f(t) = f(t)$$

$$L\left[D^{-\alpha}|Df(t)\right] = s^{-\alpha} L[Df(t)]$$

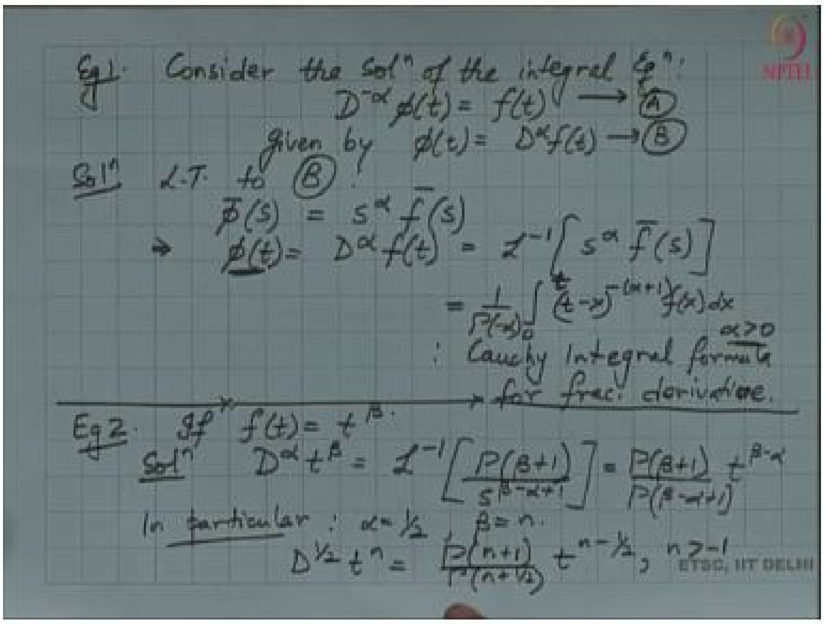
$$= s^{-\alpha}\left[s f(s) - f(0)\right]; \alpha > 0$$

Consider LT of Derivative of fractional

$$L\left[D\left(D^{-\alpha} f\right)\right] = \left\{L\left[D^{-\alpha} D f(t)\right] + f(0) L\left[\frac{t^{\alpha-1}}{t^{\alpha}}\right]\right.$$

$$= s^{-\alpha}\left[\begin{array}{c} s\bar{f}(s) - f(0) \\ +s^{\alpha} f(0) \end{array}\right]$$

$$= s^{1-\alpha} f(s); \alpha \geqslant 0$$



Example 1: Consider solution of Integral Equation:

$$D^{-\alpha}\phi(t) = f(t) \rightarrow (A)$$
$$\text{given by} \quad \phi(t) = D^{*}f(t) \rightarrow (B)$$

Solution: LT to B:

$$\phi(s) = s^{\alpha}\bar{f}(s)$$
$$\phi(t) = D^{\alpha}f(t) = L^{-1}\left[s^{\alpha}\bar{f}(s)\right]$$
$$= \frac{1}{P(x)}\int_0^t (t-x)^{-(x+1)}f(x)dx$$

Example 2:

$$\text{If } f(t) = t^{\beta}$$

Solution:

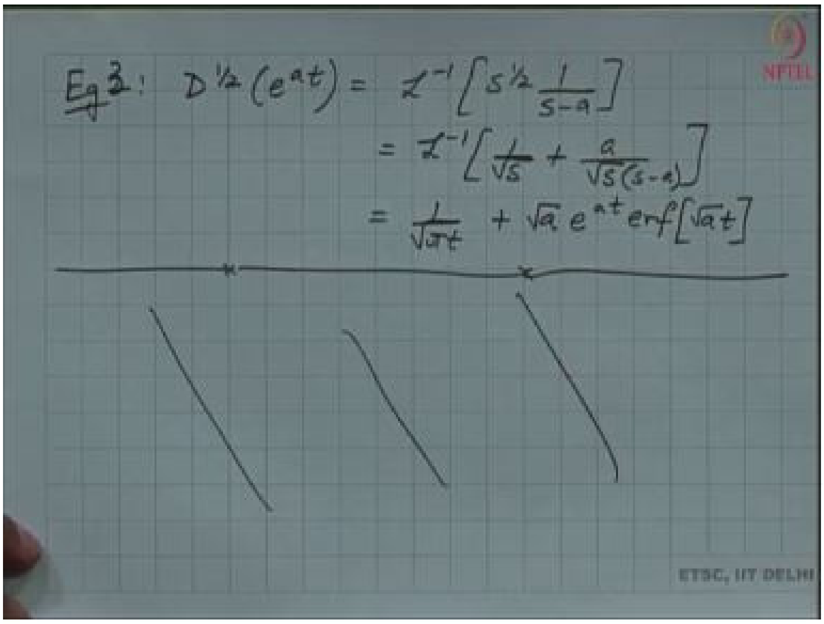
$$D^{\alpha}t^{\beta} = L^{-1}\left[\frac{P(\beta+1)}{S^{\beta-\alpha+1}}\right] = \frac{P(\beta+1)}{P(\beta-\alpha+1)}t^{\beta-\alpha}$$

In particular:

$$\alpha = 1/2, \beta = n$$

$$D^{\frac{1}{2}}t^{n} = \frac{P(n+1)}{P(n+\frac{1}{2})}t^{n-1/2}, n \geq -1$$

Example 3:



$$D^{1/2}\left(e^{at}\right) = L^{-1}\left[s^{1/2}\frac{1}{s-a}\right]$$
$$= L^{-1}\left[\frac{1}{\sqrt{s}} + \frac{a}{\sqrt{s}(s-a)}\right]$$
$$= \frac{1}{\sqrt{\pi t}} + \sqrt{a}e^{at}\,\mathrm{erf}[\sqrt{a}t]$$