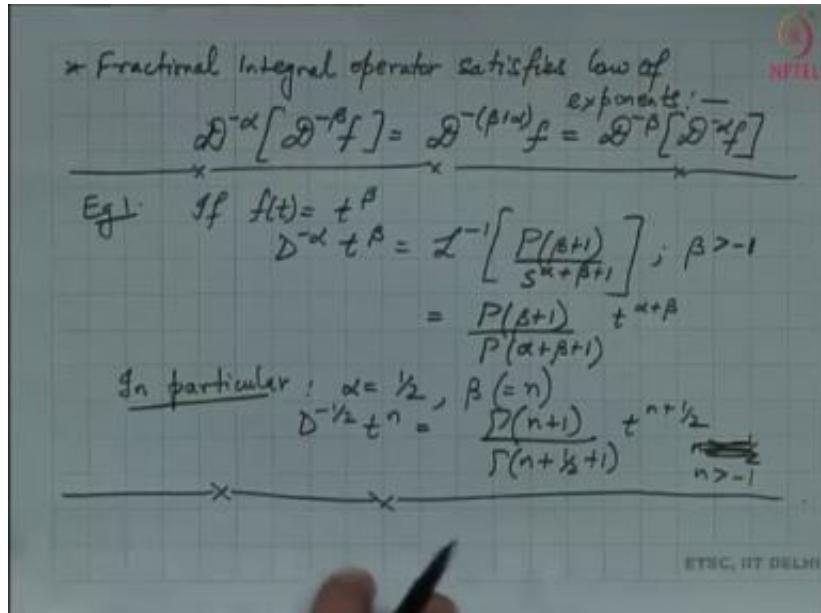


Integral Transforms and Their Applications
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Lecture -57

Introduction to Fractional Calculus Part 3



$$D^{-\alpha} [D^{-\beta} f] = D^{-(\beta+\alpha)} f = D^{-\beta} [D^{-\alpha} f]$$

Example 1:

$$\begin{aligned} \text{If } f(t) = t^\beta \\ D^{-\alpha} t^\beta = \mathcal{L}^{-1} \left[\frac{P(\beta+1)}{S^{\alpha+\beta+1}} \right]; \beta > -1 \\ = \frac{P(\beta+1)}{P(\alpha+\beta+1)} t^{\alpha+\beta} \end{aligned}$$

In particular:

$$\begin{aligned} \alpha = 1/2, \beta (= n) \\ D^{-1/2} t^n = \frac{P(n+1)}{P(n + \frac{1}{2} + 1)} t^{n+1/2} \end{aligned}$$

NPTEL

Eg 2: $\mathcal{L}^{-1}[D^{-\alpha}e^{at}]$

Solⁿ $\mathcal{L}[D^{-\alpha}e^{at}] = \frac{1}{s^\alpha(s-a)} \quad a>0$

To find $D^{-\alpha}e^{at} = \mathcal{L}^{-1}\left[\frac{1}{s^{\alpha+1}}\left\{1 + \frac{a}{s-a}\right\}\right]$

$= \frac{t^\alpha}{P(\alpha+1)} + aE(t, \alpha+1, a) \rightarrow \underline{\text{ANS}}$

where: $E(t, \alpha, a) = \frac{1}{P(\alpha)} \int_0^t \xi^{\alpha-1} e^{a(t-\xi)} d\xi$

In particular: $\alpha=1/2 \quad D^{-1/2}e^{at} = \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s}(s-a)}\right]$

$= \left\{\frac{1}{\sqrt{at}} + e^{\frac{at}{2}}\right\}$

$= \frac{e^{at}}{\sqrt{a}} \operatorname{erf}[\sqrt{at}] \rightarrow \underline{\text{ANS}}$

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Example 2:

$$D^{-\alpha}e^{at}$$

Solution:

$$\begin{aligned} L[D^{-\alpha}e^{at}] &= \frac{1}{s^\alpha(s-a)} \quad a>0 \\ D^{-\alpha}e^{at} &= L^{-1}\left[\frac{1}{s^{\alpha+1}}\left\{1 + \frac{a}{s-a}\right\}\right] \\ &= \frac{t^\alpha}{P(\alpha+1)} + aE(t, \alpha+1, a) \end{aligned}$$

where:

$$E(t, \alpha, a) = \frac{1}{P(\alpha)} \int_0^t \xi^{\alpha-1} e^{a(t-\xi)} d\xi$$

In particular:

$$\begin{aligned} \alpha = 1/2 \quad D^{-1/2}e^{at} &= L^{-1}\left[\frac{1}{\sqrt{s}(s-a)}\right] \\ &= \left\{\frac{1}{\sqrt{at}} + e^{\frac{at}{2}}\right\} \\ &= \frac{e^{at}}{\sqrt{a}} \operatorname{erf}[\sqrt{at}] \end{aligned}$$

NPTU

Eg 3: $\mathcal{L}\left[D^{-\alpha} \sin at\right]$ } Solve
 $D^{-\alpha} e^{iat}$ → Exercise

Eg 4: $\left[D^{-\alpha} \cos at\right]$ } 2 Equate real
 & Imaginary

Sol: Eg 3: $\frac{a}{s^\alpha(s^2+a^2)} ; \alpha > 0 \rightarrow \mathcal{L}^{-1}[\dots]$
 Eg 4: $\frac{s}{s^\alpha(s^2+a^2)} \rightarrow \mathcal{L}^{-1}[\dots]$

Eg 5: $\mathcal{L}\left[D^{-\alpha} [e^{at} \pm t^{\beta-1}]\right] = \frac{P(\beta)}{s^\alpha(s-a)^\beta} ; \alpha > 0, \beta > 0 \rightarrow \textcircled{1}$
 To find freq. integral: Take inverse transform
 of $\textcircled{1}$ & use convolution
 Shm.

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Example 3:

$$\{D^{-\alpha} \sin at\}$$

Example 4:

$$[D^{-\alpha} \cos at]$$

Solution: Example 3:

$$\frac{a}{s^\alpha(s^2+a^2)} \xrightarrow{\alpha>0} \mathcal{L}^{-1}[\dots]$$

Example 4:

$$\frac{s}{s^2(s^2+a^2)} \rightarrow \mathcal{L}^{-1}[\dots]$$

Example 5:

$$\mathcal{L}[D^{-\alpha} [e^{at} t^{\beta-1}]] = \frac{P(\beta)}{s^\alpha(s-a)^\beta} \xrightarrow{\beta>0}$$

(4) SPTE

NOTE: $\lim_{\alpha \rightarrow 0} {}_0D^{-\alpha} f(t) = f(t)$

* Consider LT of fractional integral of derivative of f :

$$\mathcal{L}[{}^0D^{-\alpha} [Df(t)]] = s^{-\alpha} \mathcal{L}[Df(t)] = s^{-\alpha} [sf(s) - f(0)];$$

↳ Valid for $\alpha > 0$

Consider LT of derivative of fractional int.

$$\mathcal{L}[D({}^0D^{-\alpha} f)] = \mathcal{L}[{}^0D^{-\alpha} Df(t)]$$

if $\alpha = 0$, result does not agree. {

$$+ f(0) \mathcal{L}[t^{\alpha-1}]$$

$$= s^{-\alpha} [sf(s) - f(0)] \frac{1}{\Gamma(\alpha)}$$

↳ \mathcal{L} and D do not commute $= s^{1-\alpha} f(s); \alpha \geq 0$

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$$\lim_{\alpha \rightarrow 0} D^{-\alpha} f(t) = f(t)$$

$$\begin{aligned} L[D^{-\alpha} Df(t)] &= s^{-\alpha} L[Df(t)] \\ &= s^{-\alpha} [sf(s) - f(0)]; \alpha > 0 \end{aligned}$$

Consider LT of Derivative of fractional

$$\begin{aligned} L[D(D^{-\alpha} f)] &= \{L[D^{-\alpha} Df(t)] + f(0)L\left[\frac{t^{\alpha-1}}{t^\alpha}\right] \\ &= s^{-\alpha} \left[sf(s) - f(0) \right. \\ &\quad \left. + s^\alpha f(0) \right] \\ &= s^{1-\alpha} f(s); \alpha \geq 0 \end{aligned}$$

Eg 1. Consider the Sol" of the integral Eq":

$$D^{-\alpha} \phi(t) = f(t) \rightarrow \textcircled{A}$$

Given by $\phi(t) = {}^0D^{-\alpha} f(t) \rightarrow \textcircled{B}$

Sol" L.T. to \textcircled{B} ?

$$\begin{aligned} \bar{\phi}(s) &= s^\alpha \bar{f}(s) \\ \Rightarrow \phi(t) &= {}^0D^\alpha f(t) = \mathcal{L}^{-1}[s^\alpha \bar{f}(s)] \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{-(\alpha+1)} f(x) dx \\ &\quad \alpha > 0 \\ &\quad \text{Cauchy Integral formula} \\ &\quad \text{for frac. derivative.} \end{aligned}$$

Eg 2. If $f(t) = t^\beta$.

Sol" $D^\alpha t^\beta = \mathcal{L}^{-1}\left[\frac{\Gamma(\beta+1)}{s^{\beta-\alpha+1}}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}$

In particular: $\alpha = \frac{1}{2}, \beta = n$.

$$D^{\frac{1}{2}} t^n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} t^{n-\frac{1}{2}}, n \geq 1$$

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Example 1: Consider solution of Integral Equation:

$$D^{-\alpha} \phi(t) = f(t) \rightarrow (A)$$

given by $\phi(t) = D^* f(t) \rightarrow (B)$

Solution: LT to B:

$$\begin{aligned}\phi(s) &= s^\alpha \bar{f}(s) \\ \phi(t) &= D^\alpha f(t) = L^{-1} [s^\alpha \bar{f}(s)] \\ &= \frac{1}{P(x)} \int_0^t (t-x)^{-(x+1)} f(x) dx\end{aligned}$$

Example 2:

$$\text{If } f(t) = t^\beta$$

Solution:

$$D^\alpha t^\beta = L^{-1} \left[\frac{P(\beta + 1)}{S^{\beta - \alpha + 1}} \right] = \frac{P(\beta + 1)}{P(\beta - \alpha + 1)} t^{\beta - \alpha}$$

In particular:

$$\alpha = 1/2, \beta = n$$

$$D^{\frac{1}{2}} t^n = \frac{P(n+1)}{P(n+\frac{1}{2})} t^{n-1/2}, n \geq -1$$

Example 3:

Eg 3: $D^{1/2} (e^{at}) = L^{-1} \left[s^{1/2} \frac{1}{s-a} \right]$

$$\begin{aligned}&= L^{-1} \left[\frac{1}{\sqrt{s}} + \frac{a}{\sqrt{s}(s-a)} \right] \\ &= \frac{1}{\sqrt{\pi t}} + \sqrt{a} e^{at} \operatorname{erf}[\sqrt{a}t]\end{aligned}$$

$$\begin{aligned}D^{1/2} (e^{at}) &= L^{-1} \left[s^{1/2} \frac{1}{s-a} \right] \\ &= L^{-1} \left[\frac{1}{\sqrt{s}} + \frac{a}{\sqrt{s}(s-a)} \right] \\ &= \frac{1}{\sqrt{\pi t}} + \sqrt{a} e^{at} \operatorname{erf}[\sqrt{a}t]\end{aligned}$$