Integral Transforms and Their Applications

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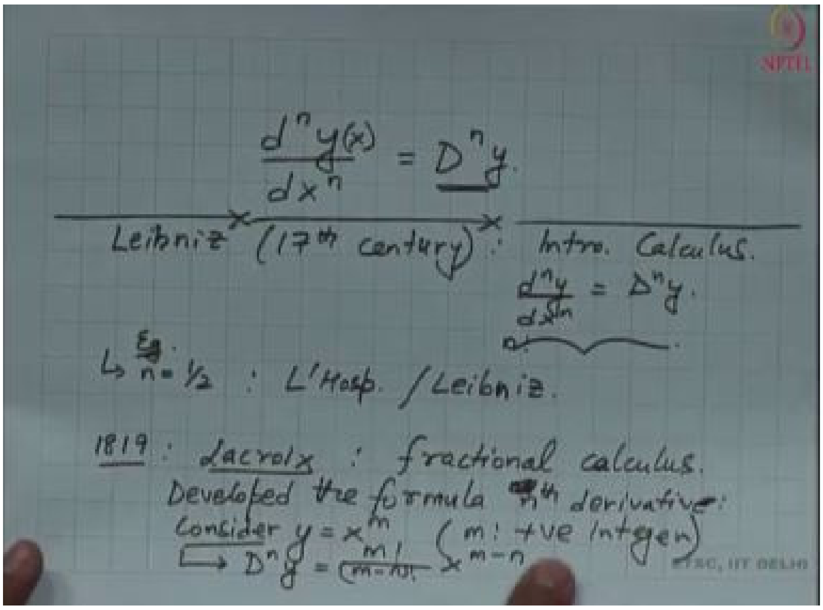
Lecture –55

Introduction to Fractional Calculus Part 1



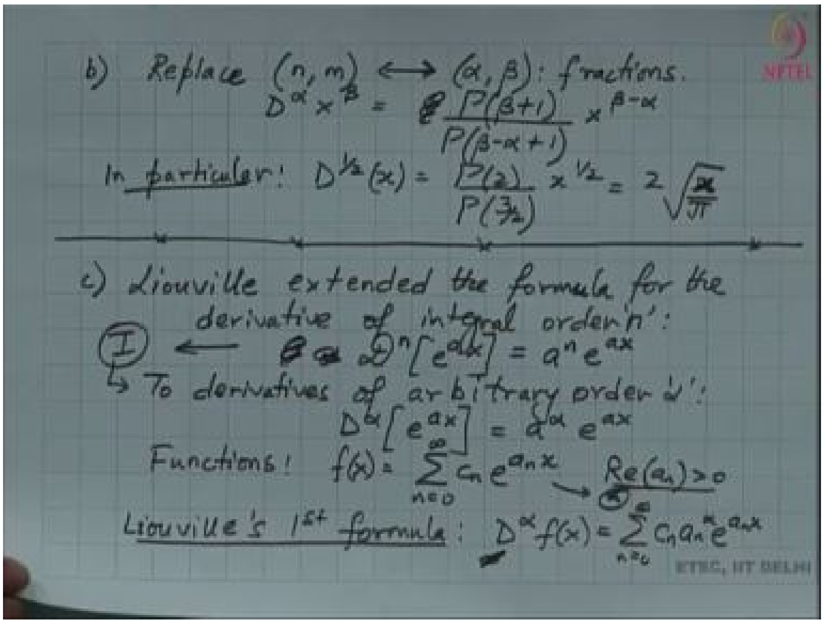
Well now, there is this important now this is a very crucial juncture where I am going to shift, slightly shift my focus and review some topics in a different sort of calculus known as the fractional calculus. And, we will see that as we look at some problems involving fractional calculus all the transforms that we have developed will be very useful in solving the problems in this area.

So, we all know how to find, we all know how to find the derivative of a function with respect to let us say an independent variable.

$$\frac{d^n y(x)}{dx^n} = \underline{D^n y}$$

Now, while discussing so it there is a legend that in 1800's there were two famous mathematicians who were discussing how to evaluate this derivative for n being negative or n being fractional. So, these two mathematicians were L'Hopital and Leibniz. So, L'Hopital asked that how to evaluate these derivatives when n is a fractional order. Now, it took some time for Leibniz to answer, but Leibniz famously quoted by saying that if I were to use a fractional order to evaluate this derivative let us say n equal to half then that will lead to some sort of a paradox. A paradox in finding the derivatives; however, he famously it is famously mentioned that he is he quoted that out of this paradox some very useful results will emerge. So, from this famous dialogue between these mathematicians it has soon from there it has soon emerged that a new calculus was developed that is the fractional calculus. So, today from the next 3, 3 to 6 lectures I am going to develop some of the basic results in fractional calculus ok. So, to start let me just highlight where did this entire discussion started. So, it was by Leibniz. So, Leibniz was a German mathematician in the 17th century who wrote a book well, he wrote an Introductory Calculus book and he talked about how to find how to find the derivatives of any function ok. So, we are very much familiar when n is integer. So, that is not a problem the problem is when n is let us say n is half. So, what is the situation and how to evaluate the derivative when n is half. So, again so this was the starting point of the emergence of fractional calculus. So, it started with the discussion by L'Hopital and Leibniz and from there Leibniz published his books that was in 17th century. Well from after this brief discussion then the subject did not pick up much, until 1819. So, this is another time point where another famous scientist by the name Lacroix he wrote another book, the book was on fractional calculus where he started to develop the formulas in evaluating these derivatives ok. So, more specifically Lacroix developed the formulas, Lacroix developed the formula for the fractional derivative using the formula for the nth order derivatives.

So, I have the formula for the nth order derivative to be given by this simple expression. So, that is quite straightforward, now from here let me just derive the famous formula by Lacroix for the fractional derivative of y.



b.

Replace $(n, m) \longleftrightarrow (\alpha, \beta)$; fractions.

$$D^{\alpha} x^{\beta} = \frac{P(\beta+1)}{P(\beta-\alpha+1)} x^{\beta-\alpha}$$

In particular:

$$D^{1/2}(x) = \frac{P(2)}{P(3/2)} x^{1/2} = 2\sqrt{\frac{x}{\pi}}$$

c. Liouville Extended formula for the derivative of integral order 'n':

$$D^{n}\left[e^{a\alpha}\right] = a^{n} e^{ax}$$

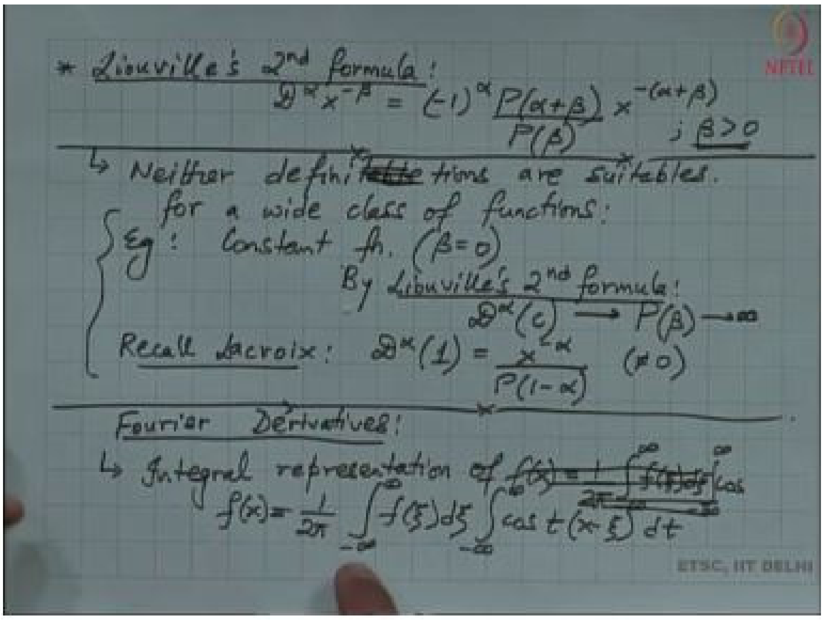
$$D^{\alpha}\left[e^{ax}\right] = a^{\alpha} e^{ax}$$

Functions:

$$f(x) = \sum_{n=0}^{\infty} c_n e^{a_n x}$$

Liouville's 1st Formula:

$$D^{\alpha} f(x) = \sum_{n=0}^{\infty} c_n a_n^k e^{a_n x}$$



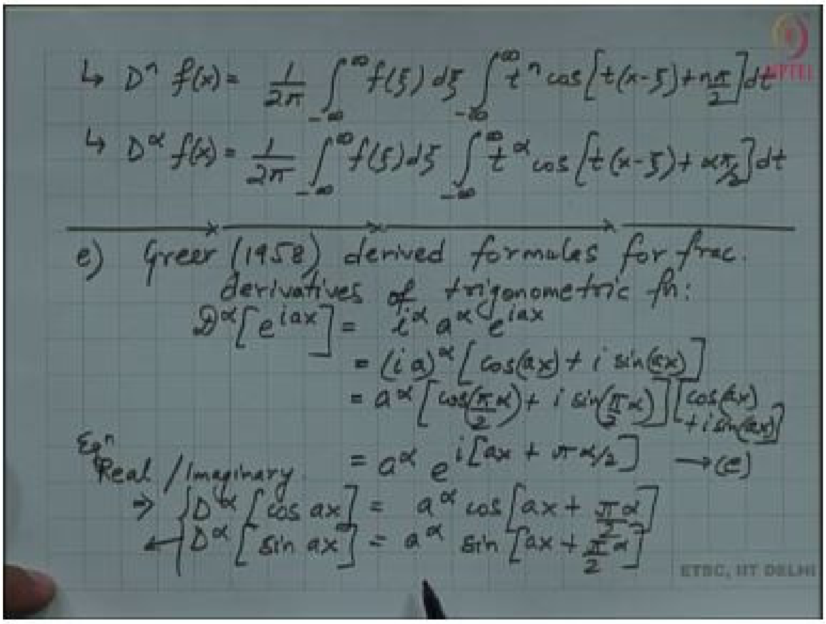
Liouville's 2nd formula:

$$D^{\alpha} x^{-\beta} = (-1)^{\alpha} \frac{P(\alpha+\beta)}{P(\beta)} x^{-(\alpha+\beta)}; \beta > 0$$

Well after introducing all this formula here is the major problem, it turns out that neither of these formulas will be completely out of error.But notice, but recall my Lacroix formula here, recall the formula that was given by Lacroix.

we are going to derive certain results, mainly derive the results on the fractional form of these functions we will see that we are mainly going to use the Liouville's expression. So, it is the Liouville's notation that we are going to follow when we derive our expressions in fractional calculus. So, then moving on let us now look at; let us now look at some Fourier derivatives. Fourier derivatives:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi) d\xi \int_{-\infty}^{\infty} \cos t(x-\xi) dt$$



$$D^n f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{-\infty}^{\infty} t^n \cos\left[t(x-\xi)+\frac{n\pi}{2}\right] dt$$

$$D^\alpha f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)d\xi \int_{-\infty}^{-\infty} t^\alpha \cos(t(x-\xi)+\alpha\pi/2]dt$$

e. Greer derived formulas for fractional derivatives of trignometric function:

$$\mathscr{D}^\alpha\left[e^{\mathrm{iax}}\right] = i^\alpha a^\alpha e^{i\alpha x}$$

$$\begin{aligned} &= (ia)^\alpha\left[\cos(\alpha x) + i\sin(\alpha x)\right] \\ &= a^\alpha\left[\cos\left(\tfrac{\pi\alpha}{2}\right) + i\sin\tfrac{(\pi\alpha)}{2}\right]\left[\begin{array}{c}\cos(\alpha x) \\ +i\sin\alpha x\end{array}\right] \end{aligned}$$

$$= a^\alpha e^{i[ax+\pi\alpha/2]}$$

Real/ Imaginary:

$$\left\{\begin{array}{l} D^\alpha[\cos ax] = a^\alpha\cos\left[ax+\frac{\pi\alpha}{2}\right] \\ D^\alpha[\sin ax] = a^\alpha\sin\left[ax+\frac{\pi\alpha}{2}\right] \end{array}\right.$$