Integral Transforms and Their Applications
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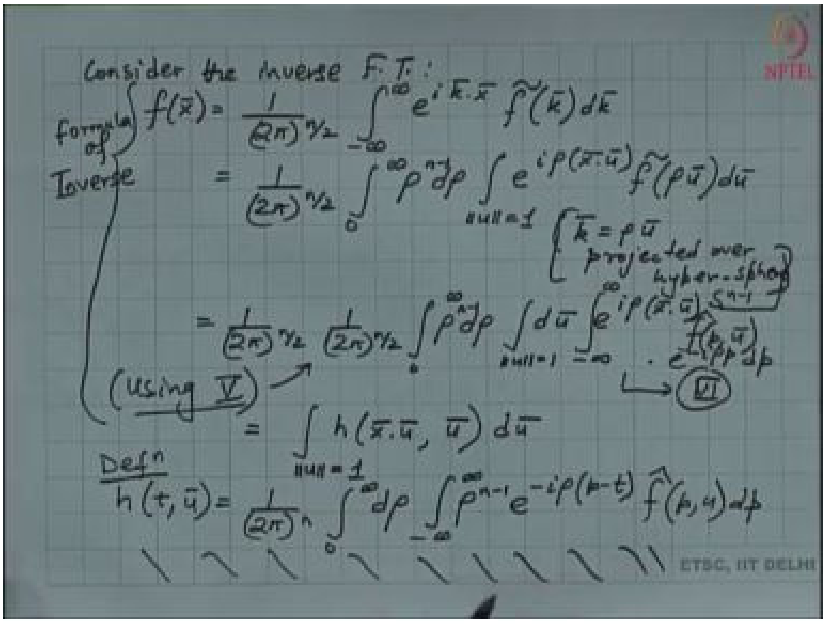
Lecture –53

Inverse Radon Transform, Applications to Radon Transform Part 2



Consider the Inverse F.T

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\bar{k}\cdot\bar{x}} \hat{f}(\bar{k}) d\bar{k}$$

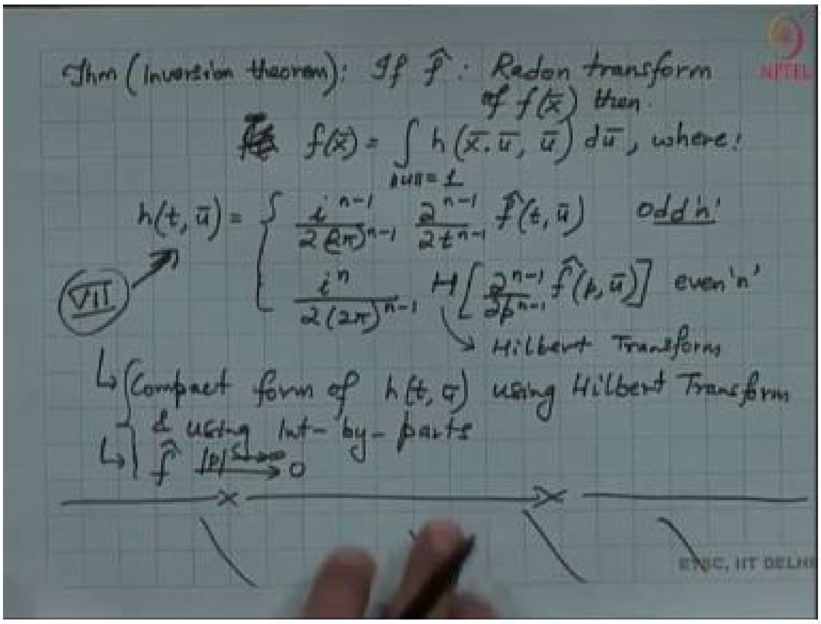
$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{0}^{\infty} \rho^{n-1} \partial\rho \int e^{i\rho(\bar{x},\bar{u})} \tilde{f}(\rho\bar{u}) d\bar{u}$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{0}^{\infty} \rho^{n-1} \partial\rho \int d\bar{u} \int_{-\infty}^{\infty} e^{ip}(\bar{x}\cdot\bar{u}) \tilde{f}(p,\bar{u}) du$$

$$= \int h(\bar{x}\cdot\bar{u},\bar{u}) d\bar{u}$$

Defn:

$$h(t,\bar{u}) = \frac{1}{(2\pi)^{n}} \int_{0}^{-\infty} d\rho \int_{-\infty}^{\infty} p^{n-1} - e^{-ip(p-t)} f(p,u) dp$$

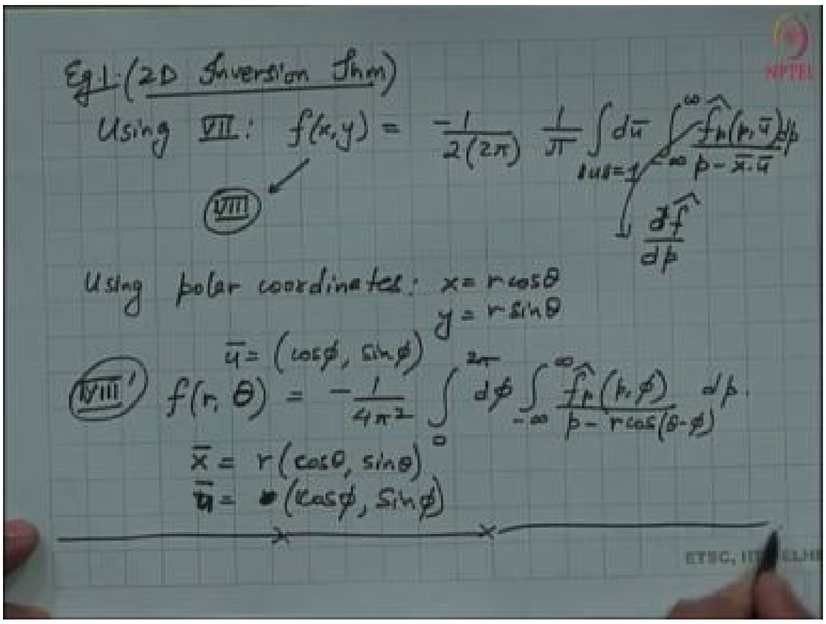


Theorem: Inverse Theorem:

If$\hat{f}$Radon transform of f then:

$$f(x) = \int h(\bar{x}\cdot\bar{u},\bar{u})d\bar{u},\ \text{where:}$$

$$h(t,\bar{u}) = \begin{cases} \frac{i^{n-1}}{2\{2\pi^{n-1}}\frac{\partial^{n-1}}{2t^{n-1}}\hat{f}(t,\bar{u}) & \text{oddn'} \\ \frac{i^n}{2(2\pi)^{n-1}} & H\left[\frac{\partial^{n-1}}{2p^{n-1}}\hat{f}(p,\bar{u})\right]\ \text{even } 'n' \end{cases}$$

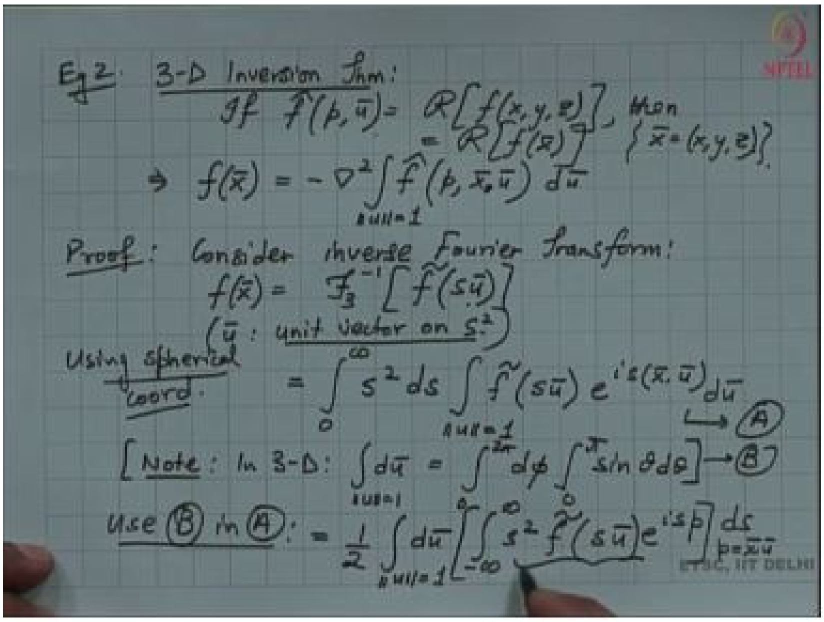


$$f(x,y) = \frac{-1}{2(2\pi)}\frac{1}{\pi}\int_{|u|=1} d\bar{u}\int_{-\infty}^{\infty}\frac{\hat{f}_p(p,\bar{u})}{p-\bar{x}\cdot\bar{u}}$$

Using polar coordinates: $x = r\cos\theta$
$y = r\sin\theta$

$$f(r,\theta) = -\frac{1}{4\pi^2}\int_0^{2\pi} d\phi \int_{-\infty}^{\infty} \frac{\hat{f}_p(p,\phi)}{p - r\cos(\theta-\phi)} dp$$

$$\bar{x} = r(\cos\theta, \sin\theta)$$
$$\bar{u} = (\cos\phi, \sin\phi)$$



Example 2: 3-D Inversion Theorem:

$$\text{If } f(p,\bar{u}) = \mathbb{R}[f(x,y,z)]\text{then}$$

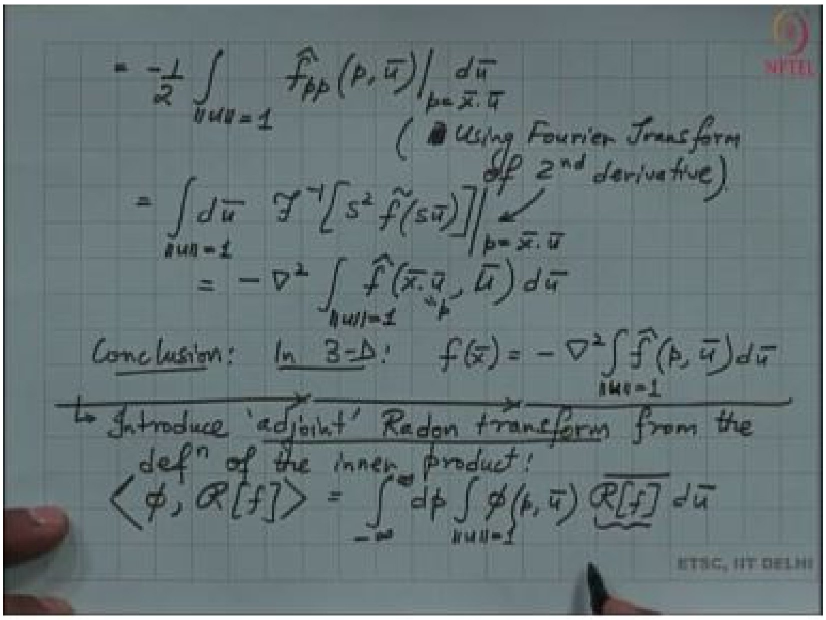
$$= R[f(x)]\{\bar{x} = (x,y,z)\}$$
$$f(\bar{x}) = f_3^{-1}[\tilde{f}(s.\bar{u})]$$

Using Spherical coordinate

$$= \int_0^{\infty} s^2 ds \int_{||u=1||} \tilde{f}(s\bar{u}) e^{is(x,\bar{u})} du$$

Note: In 3-D

$$\int_{\text{——u=1——}} d\bar{u} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta$$
$$= \frac{1}{2}\int_{||u=1||} du \left[\int_{-\infty}^{\infty} s^2 f(s\bar{u}) e^{isp}\right]_{p=\bar{x}\bar{u}} ds$$



$$= -\frac{1}{2}\int_{||u=1||} \hat{f}_{pp}(p,\bar{u})\Big|_{p=\bar{x}\cdot\bar{u}} du$$

Using Fourier Transform:

$$= \int_{||u=1||} d\bar{u} \quad \{^{-1}\left[s^2\hat{f}(su)\right]\Big|_{p=\bar{x},\bar{u}}$$

$$= -\nabla^2 \int_{||u=1||} \hat{f}(\bar{x}\cdot\bar{u},\bar{u})d\bar{u}$$

Conclusion:

$$\ln 3 - D: \quad f(\bar{x}) = -\nabla^2 \int_{||u=1||} f(p,\bar{y})d\bar{u}$$