

Integral Transforms and Their Applications
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Lecture -53

Inverse Radon Transform, Applications to Radon Transform Part 2

The image shows a handwritten derivation of the inverse Fourier transform formula. It starts with the formula $f(\bar{x}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\bar{k}\cdot\bar{x}} \hat{f}(\bar{k}) d\bar{k}$. The derivation then proceeds through several steps, including a change of variables where $\bar{k} = \rho\bar{u}$ and \bar{u} is projected over a hyper-sphere. The final result is $h(\bar{x}, \bar{u}) = \int h(\bar{x}, \bar{u}, \bar{u}) d\bar{u}$.

Consider the Inverse F.T

$$\begin{aligned}
 f(x) &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\bar{k}\cdot\bar{x}} \hat{f}(\bar{k}) d\bar{k} \\
 &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^{\infty} \rho^{n-1} d\rho \int e^{i\rho(\bar{x}, \bar{u})} \tilde{f}(\rho\bar{u}) d\bar{u} \\
 &= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_0^{\infty} \rho^{n-1} d\rho \int d\bar{u} \int_{-\infty}^{\infty} e^{ip(\bar{x}, \bar{u})} \tilde{f}(p, u) dp \\
 &\quad \xrightarrow{(IV)} \int h(\bar{x}, \bar{u}, \bar{u}) d\bar{u}
 \end{aligned}$$

Defn:

$$h(t, \bar{u}) = \frac{1}{(2\pi)^n} \int_0^{-\infty} d\rho \int_{-\infty}^{\infty} p^{n-1} - e^{-ip(p-t)} f(p, u) dp$$

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Eshm (Inversion theorem): If \hat{f} : Radon transform of $f(\bar{x})$ then:

$$\bar{f}(\bar{x}) = \int_{|u|=1} h(\bar{x} \cdot \bar{u}, \bar{u}) d\bar{u}, \text{ where:}$$

$$h(t, \bar{u}) = \begin{cases} \frac{i^{n-1}}{2(2\pi)^{n-1}} \frac{\partial^{n-1}}{\partial t^{n-1}} \hat{f}(t, \bar{u}) & \text{odd } n' \\ \frac{i^n}{2(2\pi)^{n-1}} H \left[\frac{\partial^{n-1}}{\partial p^{n-1}} \hat{f}(p, \bar{u}) \right] & \text{even } n' \end{cases}$$

→ Hilbert Transforms

↳ Compact form of $h(t, \bar{u})$ using Hilbert Transform

↳ Using Int-by-parts

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Theorem: Inverse Theorem:

If \hat{f} Radon transform of f then:

$$f(x) = \int h(\bar{x} \cdot \bar{u}, \bar{u}) d\bar{u}, \text{ where:}$$

$$h(t, \bar{u}) = \begin{cases} \frac{i^{n-1}}{2(2\pi)^{n-1}} \frac{\partial^{n-1}}{\partial t^{n-1}} \hat{f}(t, \bar{u}) & \text{odd } n' \\ \frac{i^n}{2(2\pi)^{n-1}} H \left[\frac{\partial^{n-1}}{\partial p^{n-1}} \hat{f}(p, \bar{u}) \right] & \text{even } n' \end{cases}$$

Eg1: (2D Inversion Thm)

Using VII: $f(x, y) = -\frac{1}{2(2\pi)} \frac{1}{\pi} \int_{|u|=1} d\bar{u} \int_{-\infty}^{\infty} \frac{\hat{f}_p(p, \bar{u}) dp}{p - \bar{x} \cdot \bar{u}}$

→ $\frac{d\hat{f}}{dp}$

Using polar coordinates: $x = r \cos \theta$
 $y = r \sin \theta$

(VIII') $\bar{u} = (\cos \phi, \sin \phi)$

$$\bar{f}(r, \theta) = -\frac{1}{4\pi^2} \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} \frac{\hat{f}_p(p, \phi) dp}{p - r \cos(\theta - \phi)}$$

$\bar{x} = r(\cos \theta, \sin \theta)$
 $\bar{u} = r(\cos \phi, \sin \phi)$

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$$f(x, y) = \frac{-1}{2(2\pi)} \frac{1}{\pi} \int_{|u|=1} d\bar{u} \int_{-\infty}^{\infty} \frac{\hat{f}_p(p, \bar{u})}{p - \bar{x} \cdot \bar{u}}$$

Using polar coordinates: $x = r \cos \theta$
 $y = r \sin \theta$

$$f(r, \theta) = -\frac{1}{4\pi^2} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} \frac{\hat{f}_p(p, \phi)}{p - r \cos(\theta - \phi)} dp$$

$$\bar{x} = r(\cos \theta, \sin \theta)$$

$$\bar{u} = (\cos \phi, \sin \phi)$$

Eg 2. 3-D Inversion Thm:

If $\tilde{f}(p, \bar{u}) = R[f(x, y, z)]$, then
 $\Rightarrow f(\bar{x}) = -\nabla^2 \int_{\|u\|=1} \tilde{f}(p, \bar{x}, \bar{u}) d\bar{u}$

Proof: Consider inverse Fourier transform:
 $f(\bar{x}) = F_3^{-1}[\tilde{f}(s\bar{u})]$
 $(\bar{u} : \text{unit vector on } \mathbb{S}^2)$

Using spherical coord.
 $= \int_0^\infty s^2 ds \int_{\|u=1\|} \tilde{f}(s\bar{u}) e^{is(\bar{x}, \bar{u})} d\bar{u} \quad \text{A}$

[Note: In 3-D: $\int d\bar{u} = \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \int_0^\pi d\theta$] → B

Use B in A: $= \frac{1}{2} \int_{\|u=1\|} d\bar{u} \left[\int_{-\infty}^\infty s^2 \tilde{f}(s\bar{u}) e^{isp} ds \right]_{p=\bar{x}\bar{u}}$

Example 2: 3-D Inversion Theorem:

If $f(p, \bar{u}) = R[f(x, y, z)]$ then

$$= R[f(x)]\{\bar{x} = (x, y, z)\}$$

$$f(\bar{x}) = f_3^{-1}[\tilde{f}(s\bar{u})]$$

Using Spherical coordinate

$$= \int_0^\infty s^2 ds \int_{\|u=1\|} \tilde{f}(s\bar{u}) e^{is(x, \bar{u})} du$$

Note: In 3-D

$$\begin{aligned} \int_{\|u=1\|} d\bar{u} &= \int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta \\ &= \frac{1}{2} \int_{\|u=1\|} du \left[\int_{-\infty}^\infty s^2 f(s\bar{u}) e^{isp} ds \right]_{p=\bar{x}\bar{u}} \end{aligned}$$

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$$\begin{aligned}
 &= -\frac{1}{2} \int_{||u||=1} \hat{f}_{pp}(p, \bar{u}) \Big|_{p=\bar{x}, \bar{u}} d\bar{u} \\
 &\quad (\text{Using Fourier Transform} \\
 &\quad \text{of } 2^{\text{nd}} \text{ derivative}) \\
 &= \int_{||u||=1} d\bar{u} \quad \mathcal{F}^{-1} \left[s^2 \hat{f}(su) \right] \Big|_{p=\bar{x}, \bar{u}} \\
 &= -\nabla^2 \int_{||u||=1} \hat{f}(\bar{x} \cdot \bar{u}, \bar{u}) d\bar{u}
 \end{aligned}$$

Conclusion: In 3-D: $f(\bar{x}) = -\nabla^2 \int_{||u||=1} \hat{f}(p, \bar{u}) d\bar{u}$

To introduce 'adjoint' Radon transform from the def'n of the inner product!

$$\langle \phi, \mathcal{R}[f] \rangle = \int_{-\infty}^{\infty} dp \int_{||u||=1} \phi(p, \bar{u}) \overline{\mathcal{R}[f]} d\bar{u}$$

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$$= -\frac{1}{2} \int_{||u||=1} \hat{f}_{pp}(p, \bar{u}) \Big|_{p=\bar{x}, \bar{u}} du$$

Using Fourier Transform:

$$\begin{aligned}
 &= \int_{||u||=1} d\bar{u} \quad \{^{-1} \left[s^2 \hat{f}(su) \right] \Big|_{p=\bar{x}, \bar{u}} \\
 &= -\nabla^2 \int_{||u||=1} \hat{f}(\bar{x} \cdot \bar{u}, \bar{u}) d\bar{u}
 \end{aligned}$$

Conclusion:

$$\ln 3 - D : f(\bar{x}) = -\nabla^2 \int_{||u||=1} f(p, \bar{u}) d\bar{u}$$