

Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture -52

Inverse Radon Transform, Applications to Radon Transform Part 1

The image shows a handwritten proof of the Convolution Theorem for Radon Transform. It starts with the definition of convolution $h(x) = f * g = \int_{-\infty}^{\infty} f(\bar{y})g(\bar{x} - \bar{y})d\bar{y}$, followed by the Radon transform definitions $R[f] = \hat{f}(p, \bar{u})$ and $R[g] = \hat{g}(p, \bar{u})$. The proof then shows that $R[f * g](\bar{x}) = \hat{f} * \hat{g}(\bar{x})$. The proof uses the convolution theorem for the Fourier transform and the properties of the Dirac delta function. The final result is $R[f * g](\bar{x}) = \int_{-\infty}^{\infty} f(\bar{y})g(\bar{x} - \bar{y})d\bar{y}$.

Theorem 4: Convolution Theorem for Radon Transform:

$$h(x) = \hat{f} * g = \int_{-\infty}^{\infty} f(\bar{y})g(\bar{x} - \bar{y})d\bar{y}$$

$$R[f] = \hat{f}(p, \bar{u}); R(g) = \hat{g}(p, \bar{u})$$

Proof:

$$\begin{aligned}
 \text{LHS: } R[h(x)] &= \int_{-\infty}^{\infty} h(x)\delta[p - \bar{u} \cdot \bar{x}]dx \\
 &= \int_{-\infty}^{\infty} f(\bar{y})d\bar{y} \int_{-\infty}^{\infty} g(\bar{x} - \bar{y})\delta[p - \bar{u} \cdot (\bar{x} - \bar{y})]d\bar{x} \\
 \text{Substitute } \bar{z} = \bar{x} - \bar{y} &= \int_{-\infty}^{\infty} f(\bar{y})d\bar{y} \int_{-\infty}^{\infty} g(\bar{z})\delta[p - \bar{u}(\bar{z} + \bar{y})]d\bar{z} \\
 &= \int_{-\infty}^{\infty} f(\bar{y})d\bar{y} \int_{-\infty}^{\infty} \hat{g}(p - s, \bar{u})\delta(s - \bar{u} \cdot \bar{y})ds
 \end{aligned}$$

NITTEL

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} f(y) \delta(s - \bar{u} \cdot \bar{y}) dy \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \\
 &\quad \underbrace{f(s, \bar{u})}_{\hat{f}(s, \bar{u})} \\
 &= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \int_{-\infty}^{\infty} f(y) \delta(s - \bar{u} \cdot \bar{y}) dy \\
 &\quad \underbrace{\hat{f}(s, \bar{u})}_{\hat{f}(s, \bar{u})} \\
 &= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) \hat{f}(s, \bar{u}) ds \\
 &= \hat{f} * \hat{g} = R.H.S. \quad (\text{Result}). \\
 &\xrightarrow{\text{Inverse Radon Transform!}}
 \end{aligned}$$

$\hat{f}(k)$

ETSC, IIT DELHI

$$\begin{aligned}
 &= \int_{-\infty}^{\infty} f(x) \delta(s - \bar{u} \cdot \bar{y}) dy \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \\
 &= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \int_{-\infty}^{\infty} f(y) \delta(s - \bar{u} \cdot \bar{y}) dy \\
 &= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) \hat{f}(s, \bar{u}) ds \\
 &= \bar{f} * \bar{g}
 \end{aligned}$$

Inverse Radon Transform:

$\Rightarrow \tilde{f}(\bar{k}) = \mathcal{F}[f(x)] = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\bar{k} \cdot \bar{x}} f(\bar{x}) d\bar{x}$

Consider the hyper-surface: S^{n-1} (surface of n -D hypersphere)

Use hyper-spherical coordinates: $\bar{k} = \rho \bar{u}$
 $\bar{u} \in S^{n-1}$ ($(\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^n$ s.t. $\sum_{i=1}^n |\bar{x}_i|^2 = 1$)

Ans: $\Rightarrow \tilde{f}(\rho \bar{u}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\rho(\bar{u} \cdot \bar{x})} f(\bar{x}) d\bar{x} \rightarrow \text{I}$

Fix ρ, \bar{u} : Consider $F(\bar{x}) = e^{-i\rho(\bar{u} \cdot \bar{x})} f(\bar{x})$

Radon transf. of I : $\hat{F}(\rho, \bar{u}) = \int_{-\infty}^{\infty} e^{-i\rho(\bar{u} \cdot \bar{x})} f(\bar{x}) ds \rightarrow \text{III}$

Consider $\{$ Hyperplane \mathcal{L}' : $\rho = \bar{u} \cdot \bar{x}$
 $\text{ds: element on the surface of } S^{n-1}$

ETSC, IIT DELHI

$$\tilde{f}(\bar{k}) = \mathcal{F}[f(x)] = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\bar{k} \cdot \bar{x}} f(\bar{x}) d\bar{x}$$

Consider the hyper Surface:

Use hyper spherical coordinates:

$$\bar{u} \in S^{n-1} \left((x_1, \dots, x_n) \text{ s.t. } \sum_{i=1}^n |x_i|^2 = 1 \right)$$

LHS:

$$\tilde{f}(p\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-ip(\bar{u} \cdot \bar{x})} f(\bar{x}) d\bar{x}$$

$$\text{consider } F(x) = e^{-ip(\bar{u} \cdot \bar{x})} f(\bar{x}) \rightarrow (\text{II})$$

Radon Transform of II:

$$\hat{F}(p, \bar{u}) = \int e^{-ip(\bar{u} \cdot \bar{x})} f(x) ds$$

$\hat{F}(p, \bar{u}) = e^{-ipp} \int_L f(\bar{x}) ds = e^{-ipp} \hat{f}(p, \bar{u})$

Hyper Plane $p = \bar{u} \cdot \bar{x}$

Sum over all hyper-planes!

$\Rightarrow \int_{-\infty}^{\infty} \hat{F}(p, \bar{u}) dp = \int_{-\infty}^{\infty} e^{-ipp} \hat{f}(p, \bar{u}) dp$

n-Dim Fourier Transform over hyper-surfaces!

$\tilde{f}(p\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-ip\bar{p}} \hat{f}(p, \bar{u}) dp$ (IV)

$\mathcal{F}_r[f(p\bar{u})] = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{-\infty}^{\infty} \hat{f}(p, \bar{u}) dp$ (V) Radial F.T. of

Conversely: $\hat{f}(p, \bar{u}) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{-\infty}^{\infty} \tilde{f}(p\bar{u}) \text{ Radon trans. of } f dp$ (VI)

$$\hat{F}(p, \bar{u}) = e^{-ipp} \int_L f(\bar{x}) ds = e^{-ipp} \hat{f}(p, \bar{u})$$

$$\tilde{f}(p\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-ip\bar{p}} \hat{f}(p, \bar{u}) dp$$

$$f[\tilde{f}(p\bar{u})] = \frac{1}{(2\pi)^{\frac{n-1}{2}}} f_r[\hat{f}(p, \bar{u})]$$

$$\hat{f}(p, \bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \tilde{f}(p\bar{u}) e^{ipp} dp$$