Integral Transforms and Their Applications

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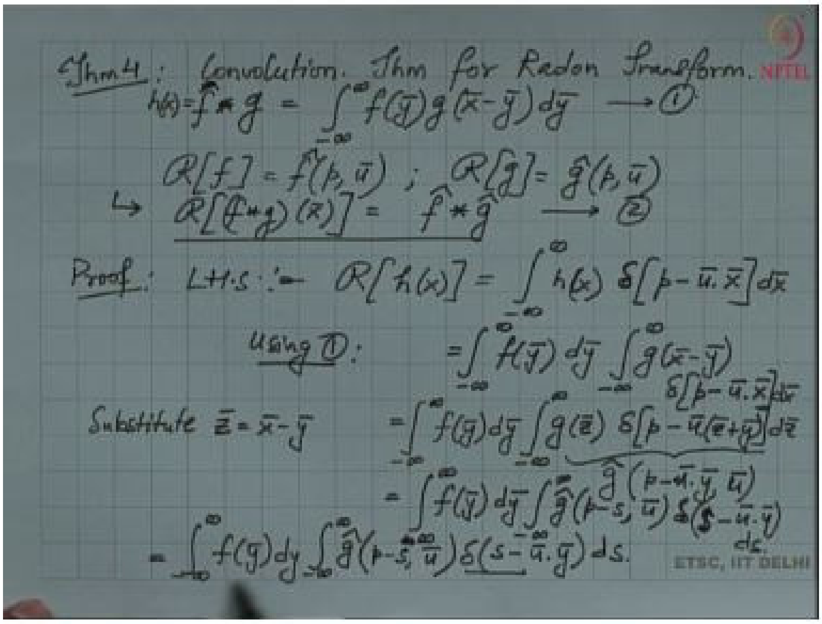
Lecture –52

Inverse Radon Transform, Applications to Radon Transform Part 1



Theorem 4: Convolution Theorem for Radon Transform:

$$h(x) = \hat{f} * g = \int_{-\infty}^{\infty} f(\bar{y}) g(\bar{x} - \bar{y}) d\bar{y}$$

$$R[f] = \hat{f}(p, \bar{u}); R(g) = \hat{g}(p, \bar{u})$$

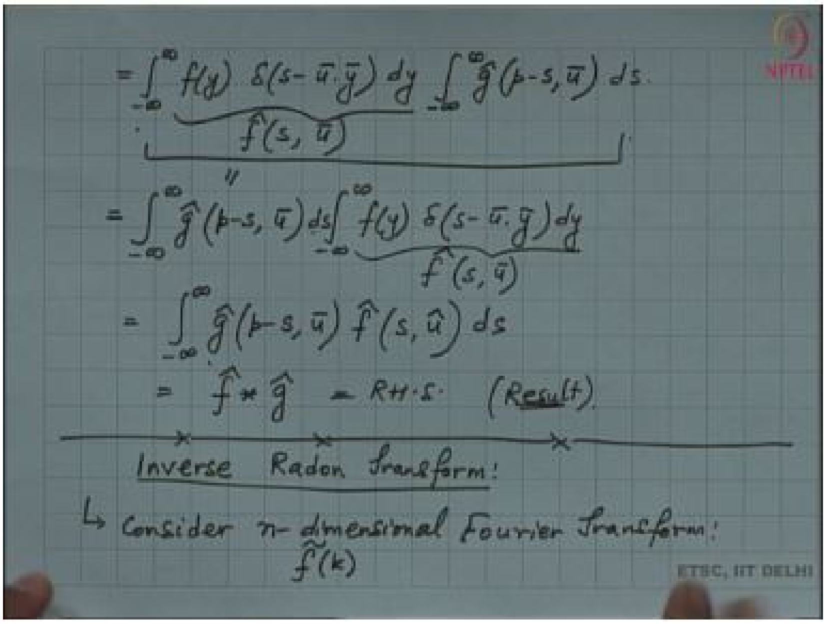
Proof:

$$\text{LHS: } R[h(x)] = \int_{-\infty}^{\infty} h(x) \delta[p - \bar{u} \cdot \bar{x}] dx$$

$$= \int_{-\infty}^{-\infty} f(\bar{y}) d\bar{y} \int_{-\infty}^{\infty} g(\bar{x} - \bar{y}) \delta[p - \bar{u} \cdot \bar{x}] dx$$

$$\text{Substitute } \bar{z} = \bar{x} - \bar{y} \quad = \int_{-\infty}^{\infty} f(\bar{y}) d\bar{y} \int_{-\infty}^{\infty} g(\bar{z}) \delta[p - \bar{u}(\bar{e} + \bar{y}] d\bar{z})$$

$$= \int_{-\infty}^{\infty} f(\bar{y}) d\bar{y} \int_{-\infty}^{\infty} \hat{g}\left(p - s, \bar{u}\right) \delta\left(s - \bar{u} \cdot \bar{y}\right) ds$$



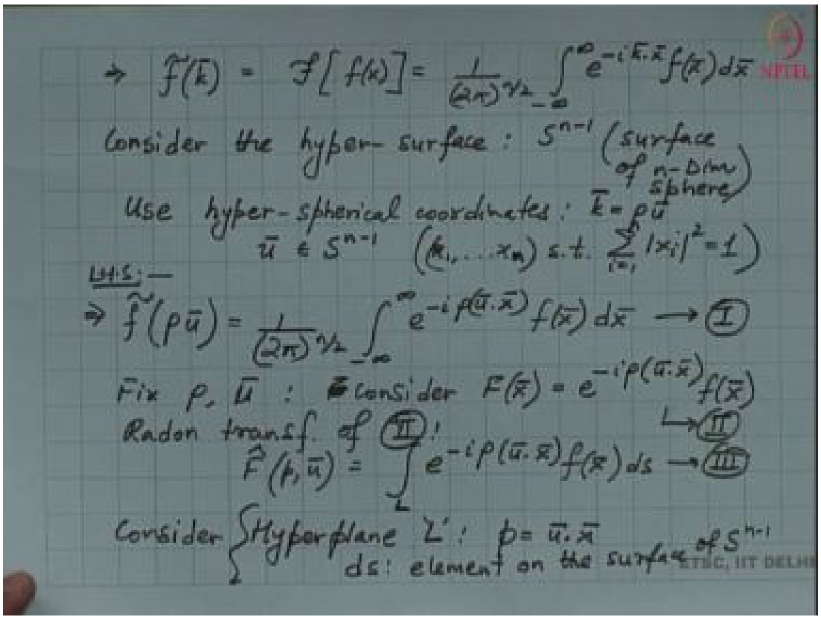
$$= \int_{-\infty}^{\infty} f(x)\delta(s - \bar{u}\cdot\bar{y})dy \int_{-\infty}^{\infty} \hat{g}(p-s,\bar{u})ds$$

$$= \int_{-\infty}^{\infty} \hat{g}(p-s,\bar{u})ds \int_{-\infty}^{\infty} f(y)\delta(s - \bar{u},\bar{y})dy$$

$$= \int_{-\infty}^{\infty} \hat{g}(p-s,\bar{u})\hat{f}(s,\hat{u})ds$$

$$= \bar{f} * \bar{g}$$

Inverse Radon Transform:



$$\tilde{f}(\bar{k}) = \mathcal{F}[f(x)] = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{-\infty}^{\infty} e^{-i\bar{k}\cdot\bar{x}} f(\bar{x})d\bar{x}$$

Consider the hyper Surface:

Use hyper spherical coordinates:

$$\bar{u} \in S^{n-1} \left((x_1, \ldots, x_n) \text{ s.t. } \sum_{i=1}^{n} |x_i|^2 = 1 \right)$$

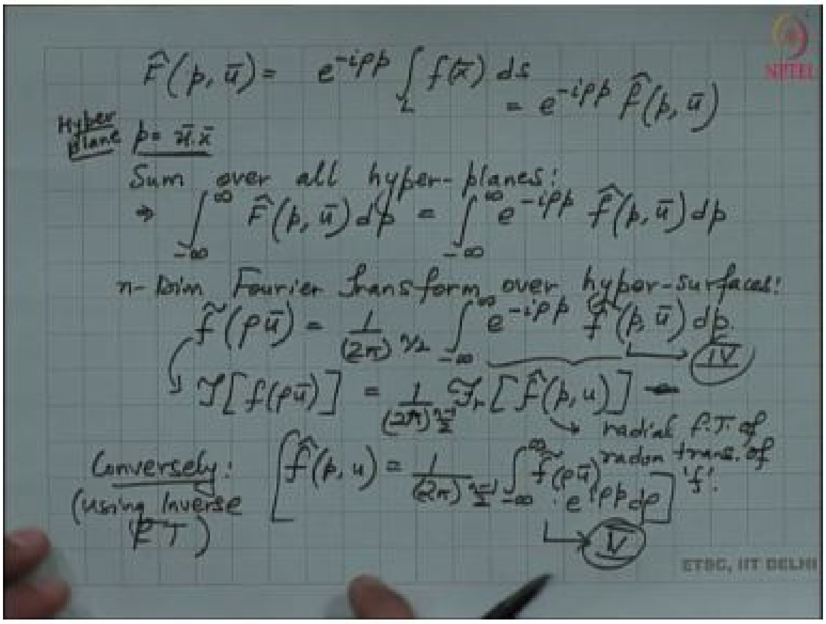
LHS:

$$\tilde{f}(p\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-ip(\bar{u}\cdot\bar{x})} f(\bar{x}) d\bar{x}$$

$$\text{consider } F(x) = e^{-ip(\bar{u}+\bar{x})} f(\bar{x}) \rightarrow \text{ (II)}$$

Radon Transform of II:

$$\hat{F}(p, \bar{u}) = \int e^{-ip(\bar{u}\cdot\bar{x})} f(x) ds$$



$$\hat{F}(p, \bar{u}) = e^{-ipp} \int_L f(\bar{x}) ds = e^{-ipp} \hat{f}(p, \bar{u})$$

$$\tilde{f}(p\bar{u}) == \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-i\rho p} \hat{f}(p, \bar{u}) dp$$

$$f[f(p\bar{u})] = \frac{1}{(2\pi)^{\frac{n-1}{2}}} f_r[\hat{f}(p, u)]$$

$$\hat{f}(p, u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \tilde{f}\,(p\bar{u})\, e^{ipp} dp$$