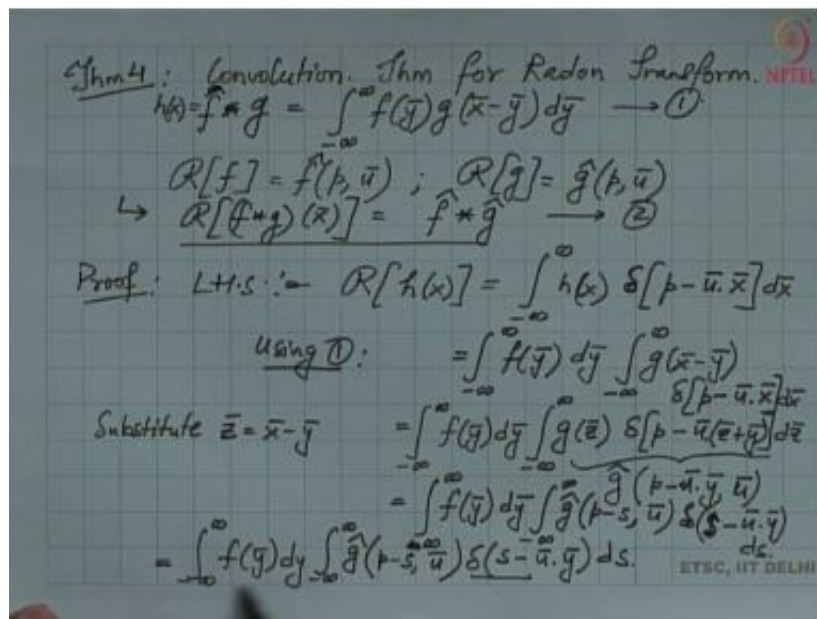


Integral Transforms and Their Applications  
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 Lecture -52

Inverse Radon Transform, Applications to Radon Transform Part 1



Theorem 4: Convolution Theorem for Radon Transform:

$$h(x) = \hat{f} * g = \int_{-\infty}^{\infty} f(\bar{y})g(\bar{x} - \bar{y})d\bar{y}$$

$$R[f] = \hat{f}(p, \bar{u}); R[g] = \hat{g}(p, \bar{u})$$

Proof:

$$\text{LHS: } R[h(x)] = \int_{-\infty}^{\infty} h(x) \delta[p - \bar{u} \cdot \bar{x}]dx$$

$$= \int_{-\infty}^{\infty} f(\bar{y})d\bar{y} \int_{-\infty}^{\infty} g(\bar{x} - \bar{y}) \delta[p - \bar{u} \cdot \bar{x}]dx$$

$$\text{Substitute } \bar{z} = \bar{x} - \bar{y} = \int_{-\infty}^{\infty} f(\bar{y})d\bar{y} \int_{-\infty}^{\infty} g(\bar{z}) \delta[p - \bar{u}(\bar{z} + \bar{y})]d\bar{z}$$

$$= \int_{-\infty}^{\infty} f(\bar{y})d\bar{y} \int_{-\infty}^{\infty} \hat{g}(p - s, \bar{u}) \delta(s - \bar{u} \cdot \bar{y}) ds$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} f(y) \delta(s - \bar{u} \cdot \bar{y}) dy \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \\
&\quad \underbrace{\hspace{10em}}_{\hat{f}(s, \bar{u})} \\
&= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \int_{-\infty}^{\infty} \underbrace{f(y) \delta(s - \bar{u} \cdot \bar{y}) dy}_{\hat{f}(s, \bar{u})} \\
&= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) \hat{f}(s, \bar{u}) ds \\
&= \hat{f} * \hat{g} = \text{R.H.S. (Result)}
\end{aligned}$$


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Inverse Radon Transform:

↳ Consider  $n$ -dimensional Fourier Transform:

$$\hat{f}(\bar{k})$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} f(x) \delta(s - \bar{u} \cdot \bar{y}) dy \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \\
&= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) ds \int_{-\infty}^{\infty} f(y) \delta(s - \bar{u} \cdot \bar{y}) dy \\
&= \int_{-\infty}^{\infty} \hat{g}(p-s, \bar{u}) \hat{f}(s, \bar{u}) ds \\
&= \bar{f} * \bar{g}
\end{aligned}$$

Inverse Radon Transform:

$$\begin{aligned}
\Rightarrow \hat{f}(\bar{k}) &= \mathcal{F}[f(x)] = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\bar{k} \cdot \bar{x}} f(\bar{x}) d\bar{x} \\
&\text{Consider the hyper-surface: } S^{n-1} \text{ (surface of } n\text{-Dim sphere)} \\
&\text{Use hyper-spherical coordinates: } \bar{k} = p\bar{u} \\
&\bar{u} \in S^{n-1} \text{ (} (x_1, \dots, x_n) \text{ s.t. } \sum_{i=1}^n |x_i|^2 = 1 \text{)} \\
&\text{L.H.S.} \Rightarrow \hat{f}(p\bar{u}) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-ip(\bar{u} \cdot \bar{x})} f(\bar{x}) d\bar{x} \rightarrow \text{I} \\
&\text{Fix } p, \bar{u} : \text{Consider } F(\bar{x}) = e^{-ip(\bar{u} \cdot \bar{x})} f(\bar{x}) \\
&\text{Radon transf. of I: } \hat{F}(p, \bar{u}) = \int e^{-ip(\bar{u} \cdot \bar{x})} f(\bar{x}) ds \rightarrow \text{II} \\
&\text{Consider Hyperplane } L: p = \bar{u} \cdot \bar{x} \text{ of } S^{n-1} \\
&\quad ds: \text{element on the surface}
\end{aligned}$$

$$\hat{f}(\bar{k}) = \mathcal{F}[f(x)] = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{-i\bar{k} \cdot \bar{x}} f(\bar{x}) d\bar{x}$$

Consider the hyper Surface:

Use hyper spherical coordinates:

$$\bar{u} \in S^{n-1} \left( (x_1, \dots, x_n) \text{ s.t. } \sum_{i=1}^n |x_i|^2 = 1 \right)$$

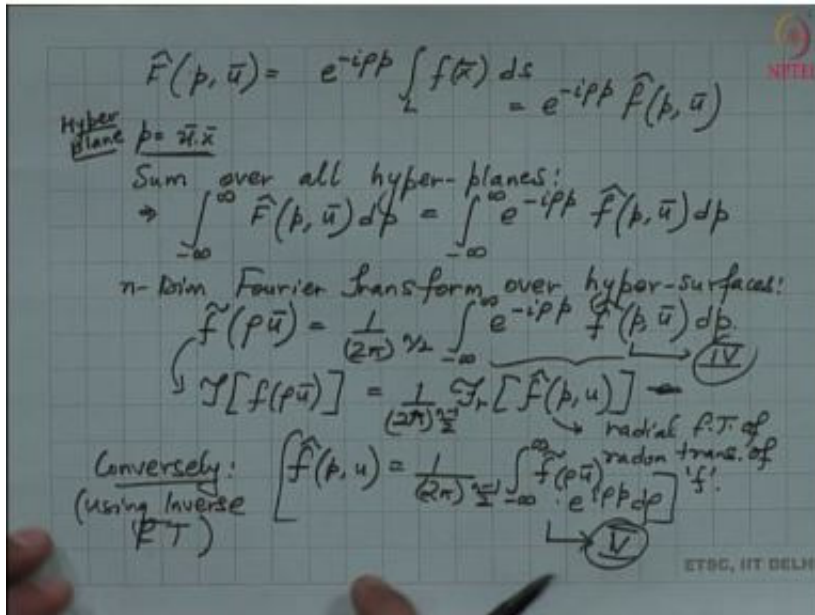
LHS:

$$\tilde{f}(p\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-ip(\bar{u}\cdot\bar{x})} f(\bar{x}) d\bar{x}$$

consider  $F(x) = e^{-ip(\bar{u}\cdot\bar{x})} f(\bar{x}) \rightarrow$  (II)

Radon Transform of II:

$$\hat{F}(p, \bar{u}) = \int e^{-ip(\bar{u}\cdot\bar{x})} f(x) ds$$



$$\hat{F}(p, \bar{u}) = e^{-ipp} \int_L f(\bar{x}) ds = e^{-ipp} \hat{f}(p, \bar{u})$$

$$\tilde{f}(p\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} e^{-ipp} \hat{f}(p, \bar{u}) dp$$

$$f[f(p\bar{u})] = \frac{1}{(2\pi)^{\frac{n-1}{2}}} f_r[\hat{f}(p, \bar{u})]$$

$$\hat{f}(p, \bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{\infty} \tilde{f}(p\bar{u}) e^{ipp} dp$$