Integral Transforms and Their Applications
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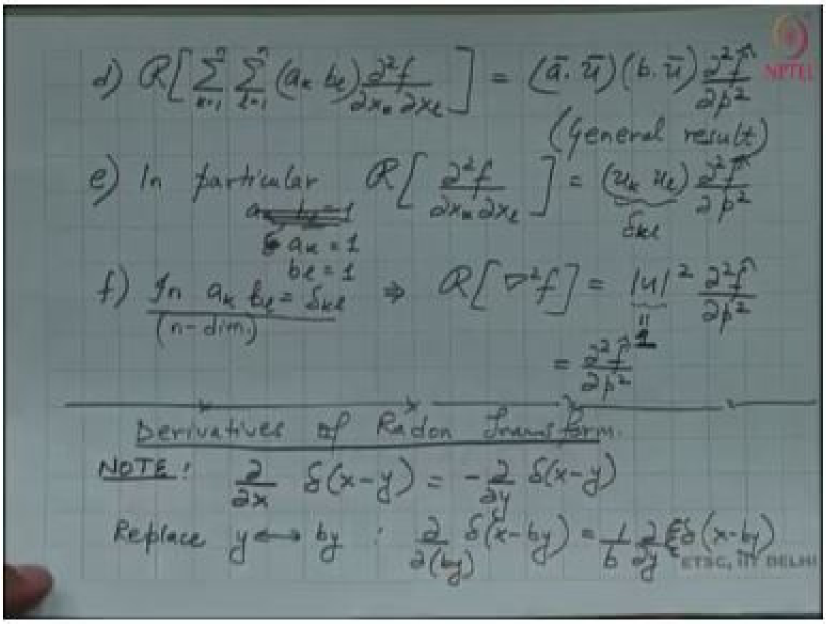
Lecture –51

Introduction to Radon Transform - Part 3



$$\text{d. } R\left[\sum_{k=1}^{n}\sum_{l=1}^{n}(a_k - b_l)\,\frac{\partial^2 f}{\partial x_k \partial x_l}\right] = (\bar{a}\cdot\bar{u})(b\cdot\bar{u})\frac{\partial^2 \hat{p}}{\partial p^2}$$

$$\text{e. In Particular, } R\left[\frac{\partial^2 f}{\partial x_k \partial x_l}\right] = (u_k u_l)\frac{\partial^2 \hat{f}}{\partial p^2}$$

$$\text{f. } R\left[\nabla^2 f\right] = |u|^2\frac{\partial^2 f}{\partial p^2}$$

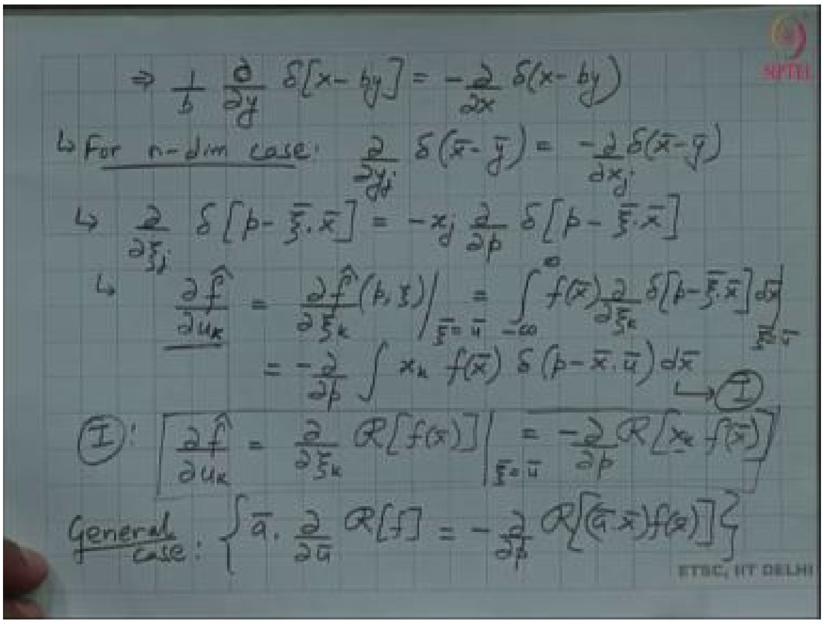
$$= \frac{\partial^2 \hat{f}}{\partial p^2}$$

Derivative of Random Transform:

$$\frac{\partial}{\partial x}\delta(x-y) = -\frac{\partial}{\partial y}\delta(x-y)$$

Replace y=by

$$\frac{\partial}{\partial (y)}\delta(x-by) = \frac{1}{b}\frac{\partial}{\partial y}(x-by)$$



$$\frac{1}{b}\frac{\partial}{\partial y}\delta[x-by] = -\frac{\partial}{\partial x}\delta(x-by)$$

For n-dim case:

$$\frac{\partial}{\partial y_j}\delta(\bar{x}-\bar{y}) = -\frac{\partial}{\partial x_j}\delta(\bar{x}-\bar{y})$$

$$\frac{\partial}{\partial \xi_i}\delta[p-\bar{\xi}\cdot\bar{x}] = -x_j\frac{\partial}{\partial p}\delta[p-\bar{\xi}\cdot\bar{x}]$$

$$\frac{\partial \hat{f}}{\partial u_k} = \frac{\partial \hat{f}}{\partial \xi_k}(\beta,\xi)\bigg|_{\bar{\xi}=\bar{u}} =$$

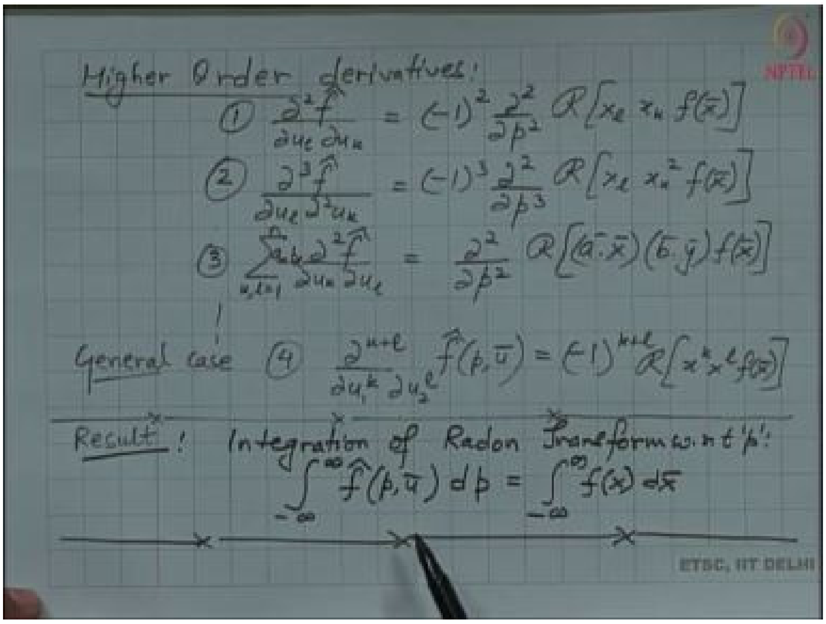
$$\int_{-\infty}^{\infty} f(x)\frac{\partial}{\partial \xi_k}\delta[p-\bar{\xi}\cdot\bar{x}]dx|_{\bar{\xi}=\bar{u}}$$

$$= -\frac{\partial}{\partial p}\int x_k f(x)\delta(p-\bar{x}\cdot\bar{u})d\bar{x}$$

$$\frac{\partial \hat{f}}{\partial u_k} = \frac{\partial}{\partial \xi_k}R[f(x)]\bigg|_{\xi=u}\frac{-\partial R}{\partial p}[x_k f(\bar{x})]$$

General Case:

$$\bar{a}\cdot\frac{\partial}{\partial \bar{u}}R[f] = -\frac{\partial}{\partial p}R[(\bar{a}\cdot\bar{x})f(\bar{x})]\Big\}$$



$$(1).\ \frac{\partial^2 f}{\partial u_c \partial u_k} = (-1)^2 \frac{\partial^2}{\partial p^2} Q\left[x_l x_k f(\bar{x})\right]$$

$$(2)\ \frac{\partial^3 f}{\partial u_l \partial^2 u_k} = (-1)^3 \frac{\partial^2}{\partial p^3} Q\left[x_1 x_2^2 f(\bar{x})\right]$$

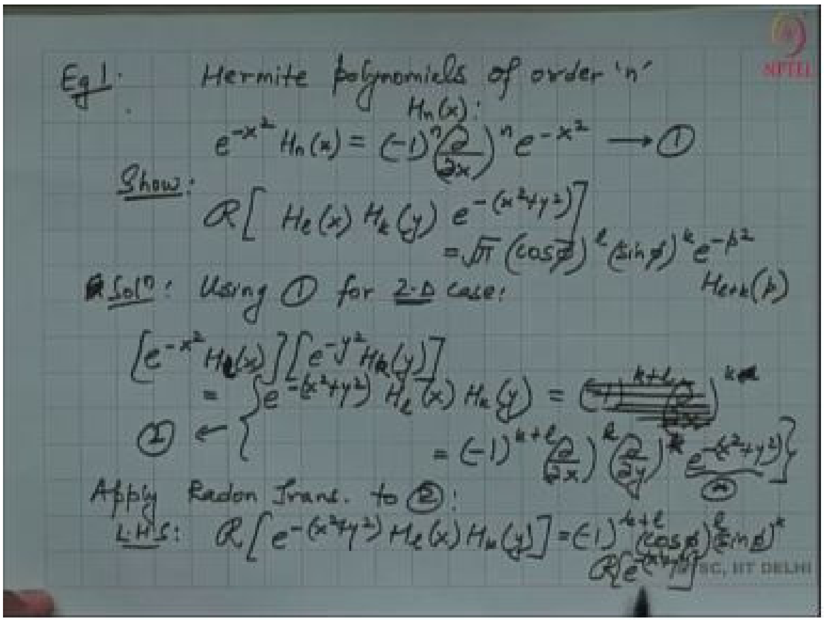
$$(3)\ \sum_{u,l=1}^{n} \frac{ab\partial^2 \hat{f}}{\partial u_k u_l} = \frac{\partial^2}{\partial p^2} R[(\bar{a}\cdot\bar{x})(\bar{b}\cdot\bar{y})f(\bar{x})]$$

$$(4)\ \frac{\partial^{k+l}}{du_1^k \partial u_2^e} \hat{f}(p,\bar{u}) = (-1)^{k+l} R\left[x^k x^l f(\bar{x})\right]$$

Result:

Integration of Random Tnansform w.r.t p

$\int_{-\infty}^{\infty} f(p,\bar{u})dp = \int_{-\infty}^{\infty} f(x)dx$



Hermite Polynomails or order n

$$e^{-x^2} H_n(x) = (-1)^n \left(\frac{\partial}{\partial x}\right)^n e^{-x^2} \to 1$$

Show:

$$\mathbb{R}\left[H_l(x) H_k(y) e^{-\left(x^2+y^2\right)}\right]$$

$$= \sqrt{\pi} (\cos\phi)^l (\sin\phi)^k e^{-p^2} H_{l+k}(p)$$

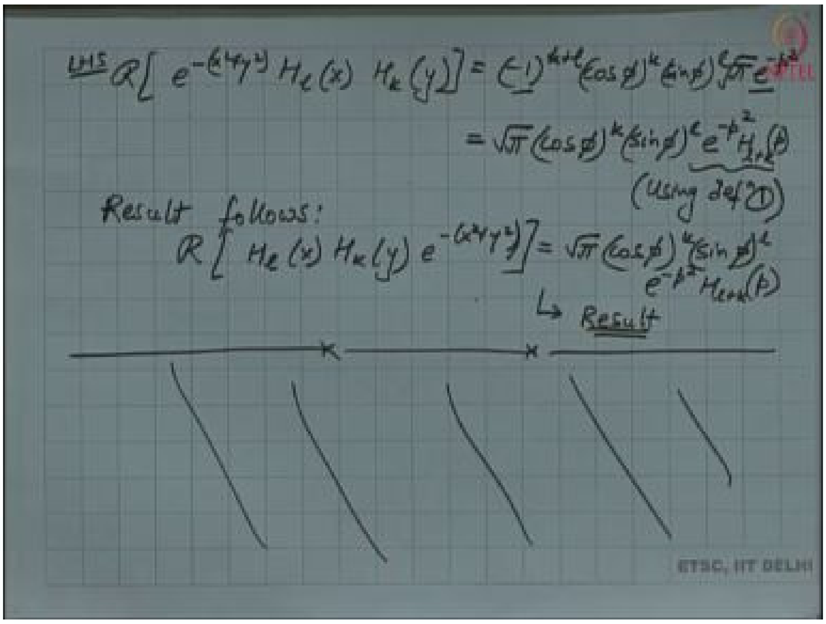
USING 1

$$\left[e^{-x^2} H_l(x)\right] \left[e^{-y^2} H_k(y)\right]$$

$$= e^{-\left(x^2+y^2\right)} H_l(x) H_k(y) =$$

$$= (-1)^{k+l} \left(\frac{\partial}{\partial x}\right)^l \left(\frac{\partial}{\partial y}\right)^k e^{-\left(x^2+y^2\right)}$$

Apply Random transform to 2:



LHS:

$$R\left[e^{-\left(x^2+y^2\right)} H_\ell(x) H_k(y)\right] =$$

$$= \sqrt{\pi} (\cos\phi)^k (\sin\phi)^l e^{-p^2} h_{l+k}(p)$$

Result:

$$R\left[H_l(x) H_k(y) e^{-\left(x^2+y^2\right)}\right] = \sqrt{\pi} (\cos\phi)^k (\sin\phi)^l e^{-p^2} h_{l+k}(p)$$