

Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture -50

Introduction to Radon Transform Part 2

The image shows a handwritten derivation of the Radon transform of $xe^{-a(x^2+y^2)}$. The steps are as follows:

$$\begin{aligned}
 & \text{Q.E.D.: } R[x e^{-a(x^2+y^2)}] \\
 &= \int_{-\infty}^{\infty} (p \cos \phi - s \sin \phi) e^{-a(p^2+s^2)} ds \\
 &= \underbrace{\int_{-\infty}^{\infty} e^{-ap^2} p \cos \phi \int_{-\infty}^{\infty} e^{-as^2} ds}_{-(\sin \phi) e^{-ap^2} \int_{-\infty}^{\infty} s e^{-as^2} ds} \quad (y = \sqrt{as}) \\
 &= \boxed{p \cos \phi e^{-ap^2} \sqrt{\frac{\pi}{a}}} \quad \underline{\text{Ans}}
 \end{aligned}$$

Properties of Radon Transform:
① Relation b/w Fourier/Radon transform.

Solution:

$$\begin{aligned}
 & R \left[x e^{-a(x^2+y^2)} \right] \\
 &= \int_{-\infty}^{\infty} (p \cos \phi - s \sin \phi) e^{-a(p^2+s^2)} ds \\
 &\quad e^{-ap^2} p \cos \phi \int_{-\infty}^{\infty} e^{-as^2} ds \\
 &\quad -(\sin \phi) e^{-ap^2} \int_{-\infty}^{\infty} s e^{-as^2} ds \\
 &\quad p \cos \phi e^{-ap^2} \sqrt{\frac{\pi}{a}}
 \end{aligned}$$

Properties of Random Transform: let us look at the most important property. The most important property of Radon transform is its relation with Fourier transform. relation between Fourier and Radon transform. let us look at the relation between Fourier and Radon transform here. So, let us see what happens.

2D Fourier Transform of 'f':

$$\hat{f}(\bar{k}) = \mathcal{F}[f(x,y)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\bar{k} \cdot \bar{x})} f(x,y) dx dy$$

④ kernel: $e^{i(\bar{k} \cdot \bar{x})} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it} \delta(t - \bar{k} \cdot \bar{x}) dt$

Using ④: $\hat{f}(\bar{k}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it} dt \int_{-\infty}^{\infty} f(\bar{x}) \delta(t - \bar{k} \cdot \bar{x}) d\bar{x}$

Let $\bar{k} = s\bar{u}$, $t = sp$ (s : real, \bar{u} : unit vector)

$$\Rightarrow \hat{f}(\bar{k}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} dp \int_{-\infty}^{\infty} sf(\bar{x}) \delta[s(p - \bar{u} \cdot \bar{x})] d\bar{x}$$

[Recall: $\int s \delta(sx) dx = \int \delta(x) dx$]

$$\Rightarrow \hat{f}(\bar{k}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} dp \int_{-\infty}^{\infty} f(\bar{x}) \delta(p - \bar{u} \cdot \bar{x}) d\bar{x}$$

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$$f(\bar{k}) = [f(x,y)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\bar{k} \cdot \bar{x})} f(x,y) dx dy$$

$$e^{i(\bar{k} \cdot \bar{x})} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it} \delta(t - \bar{k} \cdot \bar{x}) dt$$

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-it} dt \int_{-\infty}^{\infty} f(\bar{x}) \delta(t - \bar{k} \cdot \bar{x}) d\bar{x}$$

let $\bar{k} = s\bar{u}$, $t = sp$

s : real, \bar{u} : unit vector

$$f(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} dp \int_{-\infty}^{\infty} sf(x) \delta[s(p - \bar{u} \cdot \bar{x})] dx$$

$$f(\bar{k}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} dp \int_{-\infty}^{\infty} f(x) \delta(p - \bar{u} \cdot \bar{x}) dx$$

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$$\begin{aligned}
 \tilde{f}(\bar{k}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} dp \left[\underbrace{\int_{-\infty}^{\infty} f(\bar{x}) \delta[p - \bar{u} \cdot \bar{x}] d\bar{x}}_{\hat{f}(p, \bar{u})} \right] \\
 &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} \hat{f}(p, \bar{u}) dp \\
 &\quad \hookrightarrow 1\text{-D Fourier Transf. } (\hat{f}) \\
 &= \mathcal{F}[\hat{f}(p, \bar{u})] \\
 \text{2-D Fourier Trans.} &\equiv 1\text{-D Fourier Trans. (Radon Trans. off)} \\
 \text{In general: } n\text{-D Fourier Transform} & \\
 \tilde{f}(s\bar{u}) &= \frac{1}{(2\pi)^{\frac{n}{2}-1}} \mathcal{F}[R(f)] \\
 \xrightarrow{\hspace{10cm}} &\quad \xrightarrow{\hspace{10cm}}
 \end{aligned}$$

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$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isp} f(p, \bar{u}) d\beta \\
 f[\tilde{f}(p, \bar{u})]$$

In general: n-D fourier transform

$$\tilde{f}(s\bar{u}) = \frac{1}{(2\pi)^{\frac{n}{2}-1}} f[R(f)]$$

Ques 1

- ① Linearity: $R[af + bg] = aR[f] + bR[g]$
- ② Shifting: If $R[f(x, y)] = \hat{f}(p, u_1, u_2)$, then $R[f(x-a, y-b)] = \hat{f}(p - au_1 - bu_2, \bar{u})$
- n-Dim: $R[f(\bar{x} - \bar{a})] = \hat{f}(p - \bar{a} \cdot \bar{u}, \bar{u})$
- ③ Scaling: If $R[f(x, y)] = \hat{f}(p, u_1, u_2)$ then $R[af(ax, by)] = \frac{1}{|ab|} \hat{f}\left(p, \frac{u_1}{a}, \frac{u_2}{b}\right)$
- In general: $\hat{f}(a\bar{x}) = \frac{1}{|a^n|} \hat{f}\left(p, \frac{\bar{x}}{a}\right) = \frac{1}{|a^{n-1}|} \hat{f}(ap, \bar{x})$
- ④ Symmetry: If $f(p, u) = R[f(x, y)]$, if $a \neq 0$
- ⑤ $f(ap, a\bar{u}) = \frac{1}{|a|} f(p, \bar{u})$

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Theorem: 1. Linearity:

$$R[af + bg] = aR[f] + bR[g]$$

2. Shifting:

$$\text{If } R[f(x, y)] = \tilde{f}(p, u_1, u_2)$$

$$R[f(x-a, y-b)] = f(p - au_1 - bu_2, \bar{u})$$

n-dim:

$$R[f(\bar{x} - \bar{a})] = \hat{f}(p - \bar{a} \cdot \bar{u}, \bar{u})$$

3. Scaling:

$$\text{If } \mathbb{R}[f(x, y)] = f(p, u_1, u_2)$$

$$R[f(ax, by)] = \frac{1}{|ab|} f\left(p, \frac{u_1}{a}, \frac{u_2}{b}\right)$$

In general:

$$\begin{aligned}\hat{f}(a\bar{x}) &= \frac{1}{|a^n|} \hat{f}\left(p, \frac{\bar{x}}{a}\right) \\ &= \frac{1}{(a^{n-1})} \hat{f}(ap, \bar{x})\end{aligned}$$

4. Symmetry:

$$\text{If } \hat{f}(p, u) = Rf[x, y], a \neq 0$$

$$a \cdot \hat{f}(ap, a\bar{u}) = \frac{1}{|a|} f\{(p, \bar{u})\}$$

$\textcircled{b} \quad \hat{f}(p, a\bar{u}) = \frac{1}{|a|} \hat{f}\left(\frac{p}{a}, \bar{u}\right)$

Proof $\textcircled{1}$ Linearity : Exercise for students.

$\textcircled{2} \quad \mathcal{R}[f(x-a, y-b)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-a, y-b) \delta[p - xu_1 - yu_2] dx dy$

choose : $\begin{cases} \xi = x-a \\ \eta = y-b \end{cases} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta[p - (\xi+a)u_1 - (\eta+b)u_2] d\xi d\eta$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta[p - \xi u_1 - \eta u_2 - au_1 - bu_2] d\xi d\eta$

$= \hat{f}(p - au_1 - bu_2, \bar{u}) = \text{R.H.S.}$

$\textcircled{b} \quad$ General case can be proved similarly

$$b \cdot \hat{f}(p, a\bar{u}) = \frac{1}{|a|} \hat{f}\left(\frac{p}{a}, \bar{u}\right)$$

Proof: 1. Exercise for Students

2. Proof:

$$R[f(x-a, y-b)] =$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-a, y-b) \delta[p - xu_1 - yu_2] dx dy$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta[p - \xi u_1 - \eta u_2 - au_1 - bu_2] d\xi d\eta \\
&= \hat{f}(p - au_1 - bu_2, \bar{u})
\end{aligned}$$

General case can be proved similarly,

③ If $a, b > 0$; Put $ax = \xi, by = \eta$

$$R[f(ax, by)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(ax, by) \delta[p - xu_1 - yu_2] dx dy$$

Choose: $\begin{cases} \xi = ax \\ \eta = by \end{cases}$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi, \eta) \delta[p - \frac{\xi}{a}u_1 - \frac{\eta}{b}u_2] d\xi d\eta$$

$$= \frac{1}{ab} \hat{f}\left(p, \frac{u_1}{a}, \frac{u_2}{b}\right) = \underline{R+s.}$$

③b: general result follows
In particular: $a/b < 0$ $R[f(ax, by)] = \frac{-1}{ab} \hat{f}\left(p, \frac{u_1}{a}, \frac{u_2}{b}\right)$

④ Symmetry: Use $\delta[a(p - xu_1 + yu_2)] = \frac{1}{|a|} \delta(p - xu_1 - yu_2)$
Exercise to students.

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3. If $a, b > 0$; Put $ax = \xi, by = \eta$

$$\begin{aligned}
R[f(ax, by)] &= \\
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(ax, by) \delta[p - xu_1 - yu_2] dx dy \\
&= \frac{1}{ab} \hat{f}\left(p, \frac{u_1}{a}, \frac{u_2}{b}\right) \\
&a/b < 0 \quad R[f(ax, by)] \\
&= \frac{-1}{ab} f\left(b, \frac{u_1}{a}, \frac{u_2}{b}\right)
\end{aligned}$$

4. Symmetry

$$\text{Use } \delta[a(p - xu_1 + yu_2)] = \frac{1}{|a|} \delta(p - xu_1 - yu_2)$$

Exercise to students

(9) NOTE

Def^o: $\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x + \frac{h}{u_1}, y) - f(x, y)}{h}$

$R[f_x] = R\left[\frac{\partial f}{\partial x}\right] = u_1 \lim_{h \rightarrow 0} \frac{R\left[f(x + \frac{h}{u_1}, y)\right] - R[f(x, y)]}{h}$

$= u_1 \lim_{h \rightarrow 0} \frac{\hat{f}(p + h, y) - \hat{f}(p, y)}{h}$

$= u_1 \frac{\partial \hat{f}}{\partial p}(p, \bar{u})$

a. $R\left[\frac{\partial f}{\partial x}\right] = u_1 \frac{\partial}{\partial p} R[f]$ $x \leftarrow u_1$
 b. $\sim \text{Similarly: } R\left[\frac{\partial f}{\partial y}\right] = u_2 \frac{\partial}{\partial p} R(f)$ $y \leftarrow u_2$

General case: Radon Transform of 1st derivative:
 $\sim R\left[\sum_{k=1}^m a_k \frac{\partial f}{\partial x_k}\right](p, \bar{u}) = (\bar{a}, \bar{u}) \frac{\partial \hat{f}}{\partial p}$

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Derivative of Random transform:

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \left[f\left(x + \frac{h}{u_1}, y\right) - f(x, y) \right]$$

$$\begin{aligned} R[f_x] &= R\left[\frac{\partial f}{\partial x}\right] \\ &= \lim_{h \rightarrow 0} \frac{u_1}{h} \left[R\left[f\left(x + \frac{1}{u_1}, y\right)\right] - R[f(x, y)] \right] \\ &= u_1 \lim_{h \rightarrow 0} \hat{f}(p + h, y) - \hat{f}(p, y) \\ &= u_1 \frac{\partial \hat{f}}{\partial p}(p, \bar{u}) \end{aligned}$$

a.

$$R\left[\frac{\partial f}{\partial x}\right] = u_1 \frac{\partial}{\partial p} R[f]$$

Similarly, b.

$$R\left[\frac{\partial f}{\partial y}\right] = u_2 \frac{\partial}{\partial p} R(f)$$

Generally, Random Transform 1st Derivative:

$$R\left[\sum_{k=1}^m a_k \frac{\partial f}{\partial x_k}\right](p, \bar{u}) = (\bar{a}, \bar{u}) \frac{\partial \hat{f}}{\partial p}$$

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General case: $\mathcal{R}[\bar{a} \cdot \bar{\nabla} f] = (\bar{a} \cdot \bar{u}) \frac{\partial \hat{f}}{\partial p}$

Radon transform of 2nd derivatives:

- $\mathcal{R}\left[\frac{\partial^2 f}{\partial x^2}\right] = u_1 \frac{\partial}{\partial p} \left[\underbrace{\mathcal{R}[fx]}_{u_1 \frac{\partial \hat{f}}{\partial p}} \right]$
- $\mathcal{R}\left[\frac{\partial^2 f}{\partial x \partial y}\right] = u_1 u_2 \frac{\partial^2 \hat{f}}{\partial p^2}$
- If L (Linear operator) $= L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right)$
 $\mathcal{R}[Lf(x)](p, \bar{u}) = L\left[u_1 \frac{\partial}{\partial p}, u_2 \frac{\partial}{\partial p}, \dots, u_n \frac{\partial}{\partial p}\right] \hat{f}$

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General Case:

$$R[\bar{a} \cdot \bar{\nabla} f] = (\bar{a} \cdot \bar{u}) \frac{\partial \hat{f}}{\partial p}$$

Random transform of 2nd derivative:

$$\begin{aligned} \text{a. } R\left[\frac{\partial^2 f}{\partial x^2}\right] &= u_1 \frac{\partial}{\partial p} \left[\underbrace{R[fx]}_{u_1 \frac{\partial \hat{f}}{\partial p}} \right] \\ &= u_1^2 \frac{\partial^2 \hat{f}}{\partial p^2} \end{aligned}$$

$$\text{b. } R\left[\frac{\partial^2 f}{\partial x \partial y}\right] = u_1 u_2 \frac{\partial^2 \hat{f}}{\partial p^2}$$

If L (Linear operator):

$$L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right),$$

$$R[Lf(x)](p, \bar{u}) = L\left(\frac{\partial}{u_1 \partial p}, \frac{\partial}{u_2 \partial p}, \dots, \frac{\partial}{u_n \partial p}\right) \hat{f}$$