Integral Transforms and Their Applications
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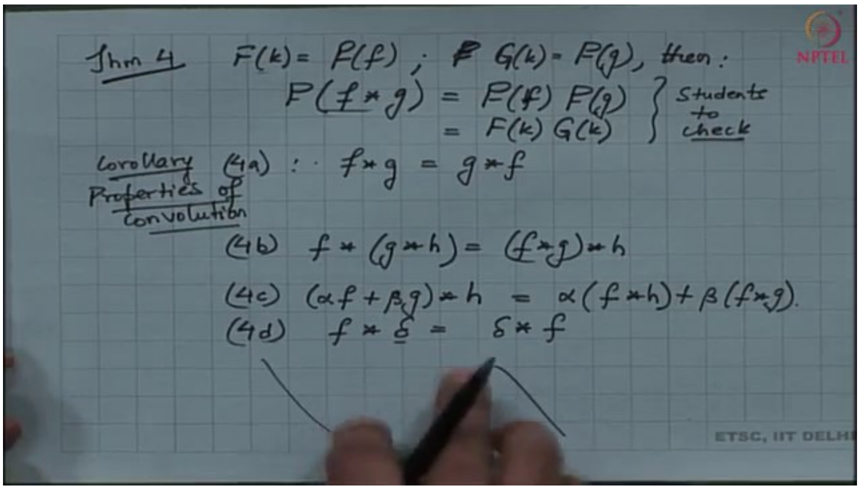
Lecture – 2
Introduction to Fourier Transforms Part - 02

So, then in this module, I am going to talk about, more results. So, let me just introduce another result in the name of another theorem, I am calling it theorem 4. It tells me that, if I have,



Theorem 4:

$$F(k) = FT(f)$$

then the result tells me, this theorem tells me that, the Fourier transform of the convolution of the 2 function is the product of the Fourier transform, the product of the Fourier transform or it is equal to

$$FT(f * g) = FT(f).FT(g) = F(k)G(k)$$

So, the Fourier transform of the convolution is quite easy to calculate, which is given by the product of the Fourier transform. Now, this is something that I would ask the students to check ok. So, this is quite easy to check; you define, you we already know what is for the convolution of 2 function then, you have to figure out, you have to apply the Fourier transform of that convolution function by use the standard definition and from there you have to separate out the 2 integrals, one corresponding to the Fourier transform of F, the other corresponding to the Fourier transform of G to arrive at this result ok.

Now, then there are some corresponding corollaries or side results, let me talk about it as 4a. So, it these are some of the properties, properties of convolution right. So, in particular one of the property says that,

So, it these are some of the properties, properties of convolution right. So, in particular one of the property says that,

$$4(a) : f * g = g * f$$

So, it does not matter the about the order of convolution then, , another property is that

$$4(b) : f * (g * h) = (f * g) * h$$

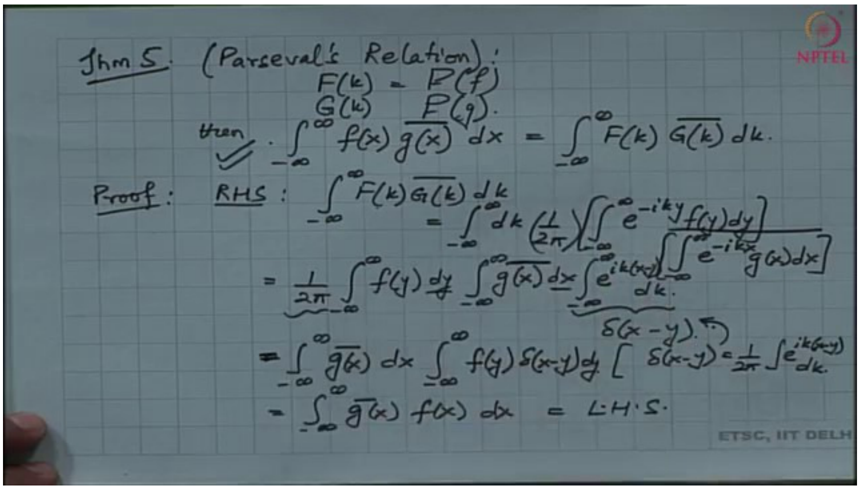
So, what this property tells us is that, it does not matter the about the order of convolution. So, you can either take this convolution first or this convolution first. So, the answer will be the same then, convolution is a linear operator right. So, in particular,

$$4\text{(c)} \ : (\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h)$$

So, it is a linear operator and finally, another important result is that the convolution of the delta function is the delta function. So, the delta function also obeys are corollary 4a. I introduce this last result separately because; delta function is not a function in the classical sense, as outlined it in my earlier lecture.

$$4 \text{ (d)} : \quad f * \delta = \delta * f$$

So, then I have one more result, which I am going to introduce in the form of another theorem, called the Parseval's relation; Parseval's relation.



So, what is Parseval's relation? Well, Parseval's relations are quite useful when we were; we have to take the inner product of 2 functions or suppose we were to define the Fourier transform of the product of 2 functions then Parseval's relations are quite useful, in those sort of evaluations. So, let us see what is this Parseval's relation.

$$F(k) = FT(f)$$

$$G(k) = FT(g)$$

then,

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} F(k)\overline{G(k)}dk$$

So, let me show you the proof quickly because, this is quite an important result. So, I am going to start with my right hand side RHS. So, my RHS is:

$$\int_{-\infty}^{\infty} F(k)\overline{G(k)}dk = \int dk \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} e^{-iky} f(y)dy \int_{-\infty}^{\infty} e^{-ikx} g(x)dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y)dy \int_{-\infty}^{\infty} \overline{g(x)}dx \int_{-\infty}^{\infty} e^{ik(x-y)}dk$$

$$= \int_{-\infty}^{\infty} \overline{g(x)} dx \int_{-\infty}^{\infty} f(y) \cdot \delta(x-y) dy$$

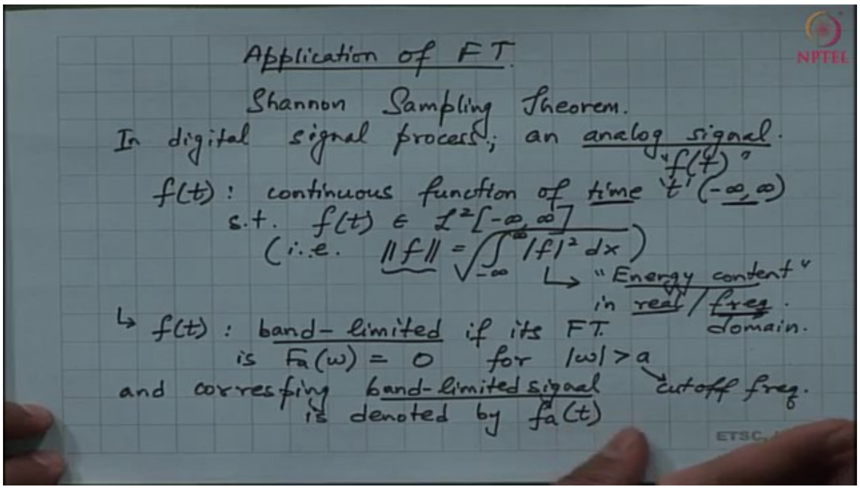
$$\text{where, } \delta(x-y) = \frac{1}{2\pi} \int e^{ik(x-y)} dk$$

$$= \int_{-\infty}^{\infty} \overline{g(x)} f(x) dx = L \cdot H \cdot S$$

So, finally, I am left with the above result, and that is my left hand side of my Parseval's relation here. So, you see that the inner product or the inner product of the two functions in the physical plane, is the inner product of the two function in the transformed plane right and they are equal.

Application of Fourier Transform:

Then I am going to introduce, a popular exam; a popular application, a popular application of Fourier transform namely the Shannon sampling theorem.



So, before I move on to describe what is the Shannon sampling theorem; I just want to highlight that, this theorem this particular result is a very important bridge between connecting continuous functions and discontinuous or discrete functions. Specially in the area of signal processing and communication; Shannon functions or Shannon sampling theorem plays a major result, which helps us to connect discrete signals into continuous time continuous signals.

So, I will start today with the properties of Fourier transform. So, let me denote my Fourier transform of a function f as F(k). So, my k is the transformed variable and the transformed function for which I am evaluating the Fourier transform is small f. So, then some of the properties that are useful are I am going to prove and show some of these properties. So, let us see what is this Shannon transform or Shannon sampling theorem. So, as I said in digital signaling; in digital signal processing so I am just going to build up some useful terms here. So, in digital signal process processing, I have an analog signal, I have an analog signal, which is denoted by f(t) right.So, f(t) is a continuous function of let us say my variable is time t and I am going to take well, I am going to take without loss of generality from negative infinity to infinity. You will see why this choice is taken.

So, it is; it may seem to be a bit counter intuitive to take this limit for the time t. So, I have; so f(t) is an analog function, which is a continuous function, such that I have f(t), I need

$$f(t) \in L^2[-\infty, \infty]$$

So, which means, when I say f(t) is L^2 integrable, I say I am what I mean by that is that, that the:

$$\|f\| = \sqrt{\int_{-\infty}^{\infty} |f|^2 dx}$$

So, so what do I mean by that? So, suppose you have wide spectrum. So, suppose you are trying to represent your information with this analog signal and we want to see how much energy or how much you know what is the amplitude of this signal, for all such possible frequencies k. So, what we do is we find the norm of these function and that is that gives us the total energy content of the signals that are being transferred. So, that is what I mean by this energy content.

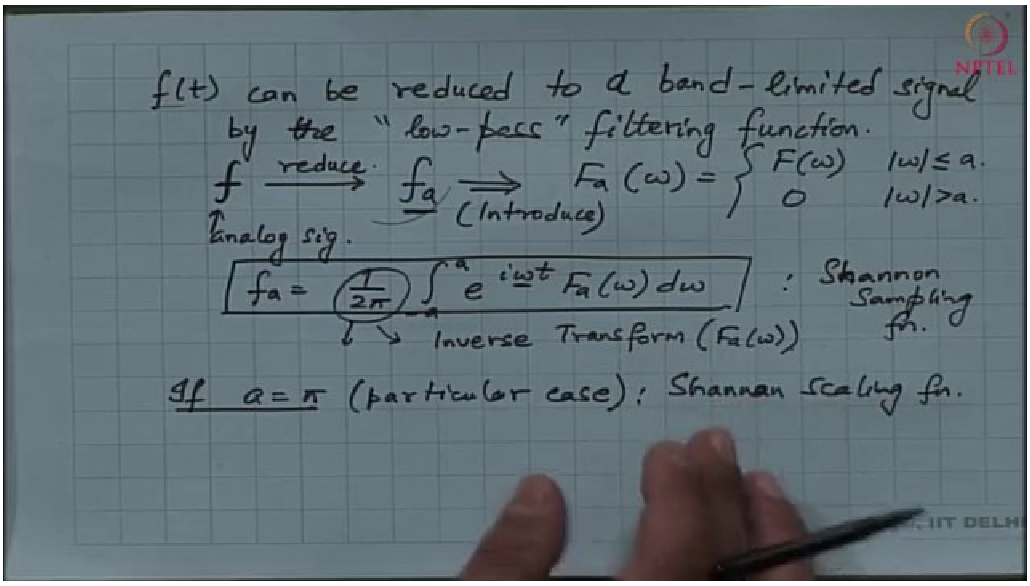
So, it could be either in well; real or the frequency domain. Well, why because they are the same in the real and the frequency domain because of our Parseval result; the Parseval theorem that we had proved earlier. Now, let me introduce another definition.

If its Fourier transform is given by :

$$F_a(\omega) = 0 \quad \text{for} \quad |\omega| > a$$

where $F_a(\omega)$ a of omega is defined to be 0 for all frequencies which are bigger than some cutoff frequency a.

So, I call this a as my cutoff frequency. So, you can see that, if the signal is band limited, then it is non zero only for a particular range of frequency, outside that range the frequency will be; the signal will be 0 Then the corresponding signal is; the corresponding signal the band-limited signal, the corresponding band-limited signal is denoted by $f_a(t)$ is right. So, so I call this as my band-limited signal, if my Fourier transform is 0 outside a particular cutoff frequency. So, in general, I can always reduce.



So, f(t), if f(t) is not band-limited, I can always reduce f(t) to a band-limited signal, how? Notice; so f(t) can always be reduced to a band-limited signal, by the so called so called low-pass filter, low-pass filter; filtering. So, again what is low-pass filter; that is another function right. So, what is this function? So, if I have a analog signal f right. So, this is my analog signal, analog signal, which is that time continuous function and we want to reduce it into a band limited signal, we want to reduce this.

So, I am going to introduce for that, I am going to introduce, I am going to introduce, my band, I am going to introduce my Fourier transformed function $F_a(\omega)$ which is your usual Fourier transform inside this range of frequency and 0 otherwise. So, what I am trying to do is, if I have an analog signal and I want to reduce it to a band-limited signal; I introduce its Fourier transform, such that it is 0 outside a particular domain. So, then of course, when we take an inverse transform, I get that the signal is the band- limited signal.

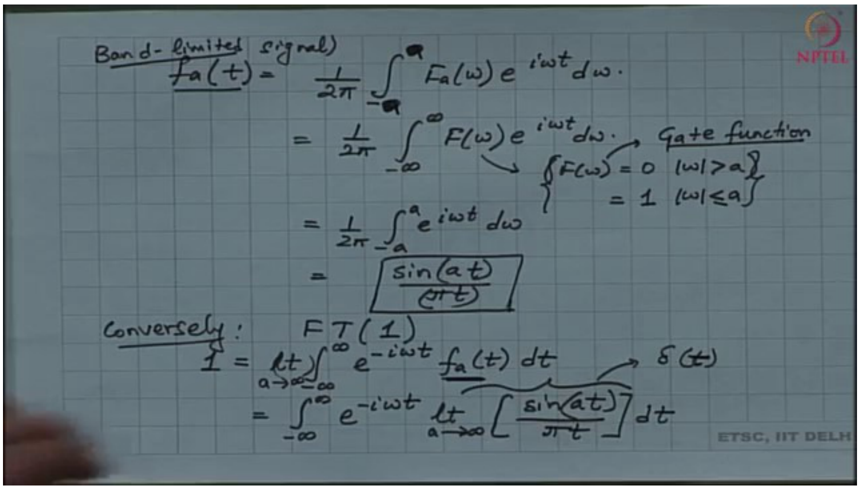
$$f \overset{\text{reduce}}{\longrightarrow} f_a \Rightarrow F_a(\omega) = \begin{cases} F(\omega) & |\omega| \leq a \\ 0 & |\omega| > a \end{cases}$$

So, so then what is the band-limited signal? The band limited signal is given by:

$$f_a = \left(\frac{1}{2\pi}\right) \int_{-a}^{a} e^{i\omega t} F_a(\omega) dw$$

Now, notice that I have used a factor of 2 pi, not 1 by 2 pi, 1 by 2π not 1 by $\sqrt{2\pi}$. So, the fact is that, we are using the definition used by the electrical engineers that was introduced in my lecture 1. So, I am introducing, you know angular frequency instead of my wave number. So, that is another definition of the Fourier inverse and the Fourier transforms.

So, for this band-limited signal, defined by this result, this is also called as my Shannon sampling function; Shannon sampling function. So, if I have that my a is π, in that particular case. So, that is the particular case then I have the Shannon scaling function, Shannon scaling function. So, f is my Shannon sampling function. So, I am going to use this result, to show what is my Shannon theorem.



So, then well; so again so, I have my band-limited signal. So, this is my band-limited signal, band-limited signal, $f_a(t)$ defined as:

$$f_a(t) = \frac{1}{2\pi}\int_{-a}^{a} F_a(w)e^{i\omega t}dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$$

$$\text{where, } F(w) = \begin{Bmatrix} 0 & |w| > a \\ 1 & |\omega| \leqslant a \end{Bmatrix}.$$

So, I call this as my gate function, my gate function. So, in a gate function, the Fourier transformed are transformed functions are specifically defined inside the cutoff range and outside the cutoff range.So in this case on further evaluating we get,

$$= \frac{1}{2\pi}\int_{-a}^{a} e^{i\omega t}d\omega$$

$$= \frac{\sin(at)}{(\pi t)}$$

So, I get that for a gate function, my band-limited signal is given by this, this following closed form expression sin a t by pi t ok. So conversely; so, conversely; if I have, look at the Fourier transform of this constant 1. So, so the Fourier transform of 1 is given by :

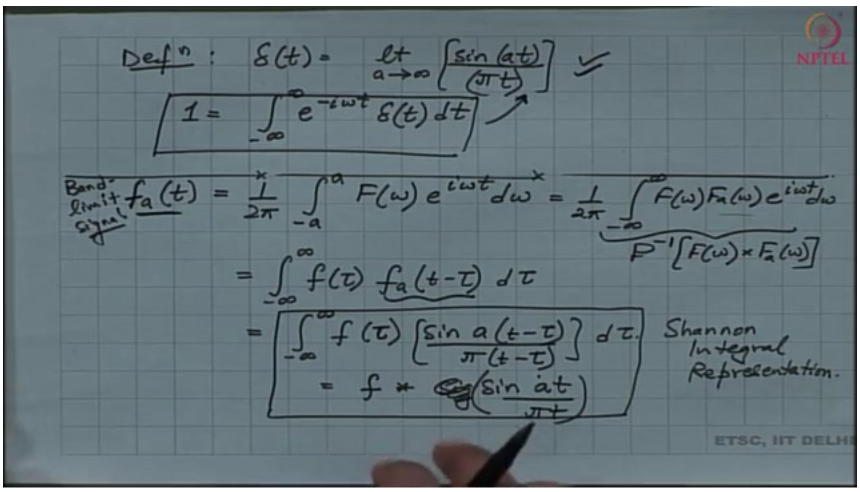
$$\text{FT}(1) = \lim_{a\to\infty}\int_{-\infty}^{\infty} e^{-i\omega t}f_a(t)dt$$

$$= \int_{-\infty}^{\infty} e^{-i\omega t}\lim_{a\to\infty}\left[\frac{\sin(at)}{\pi t}\right]$$

Now, what I have is the following:

$$\delta(t) = \lim_{a\to\infty}\left[\frac{\sin(at)}{(\pi t)}\right]$$

So,this is nothing but delta function. This is another definition of delta function.



If it is not intuitively clear, you students should plot this function and allow this parameter a to go to infinity and see that it approaches delta function. So, then coming back, I have 1 is equal to:

$$1 = \int_{-\infty}^{\infty} e^{-i\omega t}\delta(t)dt$$

so what I am left with this that well; so we have the following result, which brings us to this definition as well. So, then I have; then I have the following.

So, let me continue; so, I have the that the band-limited signal:

$$f_a(t) = \frac{1}{2\pi}\int_{-a}^{a} F(w)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)F_a(\omega)e^{i\omega t}d\omega$$

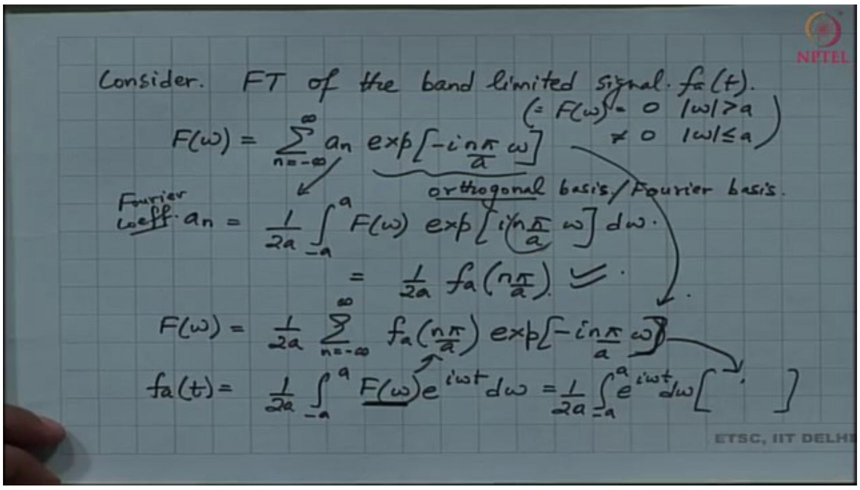
. So, we see that, we see that this is nothing but, the Fourier transform inverse of the product of two transform functions,$F(\omega)$ times, $F_a(\omega)$. So, this is the Fourier inverse of the product of two transforms, this particular integral and I get that.

$$= \int_{-\infty}^{\infty} f(\tau)f_a(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} f(\tau)\left[\frac{\sin a(t-\tau)}{\pi(t-\tau)}\right]d\tau$$

$$= f * \left(\frac{\sin at}{\pi t}\right)$$

So, this is the convolution. So, my band-limited signal is a convolution of your original signal with this factor. So, I call this result, I call this representation as my Shannon integral representation. So, I am going to end this module and continue to the next module, describing what is this Shannon integral representation and how does it connect with our Shannon sampling theorem.



So, let us consider, let us consider the Fourier transform, the Fourier transform of the band-limited signal, fa of t. So, then I know that

$$\begin{cases} F(w) = 0 & |\omega| > a\} \\ \neq 0 & |\omega| \leqslant a\} \end{cases}$$

So, that is my band Fourier transform of my band-limited signal. So, I can always write; I can always write my Fourier transform, in terms of this series, in terms of this orthogonal sets,

$$F(\omega) = \sum_{n=-\infty}^{\infty} a_n \exp\left[-i\frac{n\pi}{a}\omega\right]$$

So, I know that, these are my; so these are my orthogonal sets right. So, these are my orthogonal basis functions or the Fourier basis, I call this as my Fourier basis. So, I can always represent my Fourier transform in terms of this relation and of course, my Fourier coefficients, my Fourier coefficients, a_n is given by:

$$= \frac{1}{2a}\int_{-a}^{a} F(w)\exp\left[-i\frac{n\pi}{a}\omega\right] d\omega$$

$$= \frac{1}{2a} f_a\left(\frac{n\pi}{a}\right)$$

So, if I replace this by t, then I get that this integral is equal to fa of t. So, this is; so the coefficients are 1 by 2 a, fa evaluated at this argument. So, I am going to use this result.

$$F(w) = \frac{1}{2a}\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\exp\left[-i\frac{n\pi}{a}\omega\right]$$

So, my $F(\omega)$, the Fourier transform is:

$$F(w) = \frac{1}{2a}\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\exp\left[-i\frac{n\pi}{a}\omega\right]$$

So, I have, I have just used this relation here.

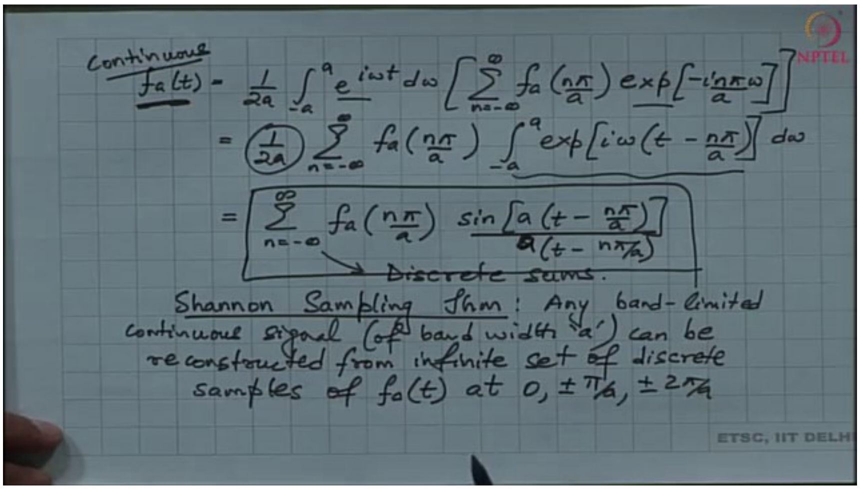
So, what I see is, let us just keep looking at this expression, what I have is, that this is also equal to;

$$f_a(t) = \frac{1}{2a}\int_{-a}^{a} F(\omega)e^{i\omega t}d\omega = \frac{1}{2a}\int_{-a}^{a} e^{i\omega t}d\omega$$

$$f_a(t) = \frac{1}{2a}\int_{-a}^{a} e^{i\omega t}dw\left[\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\exp\left[-\frac{in\pi w}{a}\right]\right]$$

$$= \frac{1}{2a}\left[\sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\right]\int_{-a}^{a}\exp\left[i\omega\left(t-\frac{n\pi}{a}\right)\right]d\omega$$

$$= \sum_{n=-\infty}^{\infty} f_a\left(\frac{n\pi}{a}\right)\frac{\sin\left[a\left(t-\frac{n\pi}{a}\right)\right]}{a(t-\frac{n\pi}{a})}$$

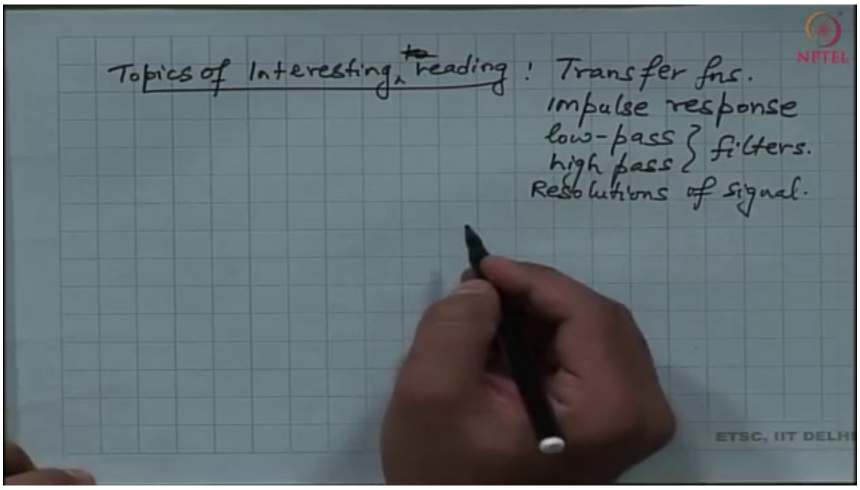


So, what I am getting is that f a the band, this is a continuous, this is a continuous function, this is a continuous signal and these are these are discrete sums.

So, this is the; this is a result that I was trying to arrive at, also known as the Shannon sampling theorem and what it says is that, I can represent, I can represent my continuous function by an infinite sum of discrete functions or if I have some discrete signals, I can construct my continuous signal by the following summation of the discrete signals, that is the power of this theorem, the Shannon sampling theorem. So, specifically it tells us that, any band-limited so let me write down the statement of this theorem.

So, any band-limited continuous signal of let us say bandwidth a, can be reconstructed from infinite set of discrete samples, infinite set of discrete samples of fa, at 0, at all these points, you know integral multiples of pi by a. So, this is what the result shows us. So, you can construct, you can construct a continuous signal out of the infinite sum of discrete signal and that is what the theorem says.

So, I want to end this discussion by providing few more topics, topics of interesting well; topics of interesting reading. So, people who are specially working in this area of low-pass filtering, band pass filtering or Shannon sampling theorem, they should also look at topics like transfer functions, what are these impulse response functions.



So, these interestingly all these ideas, all these notions could be very easily described by our Fourier transformed theory that we have just developed. So, low pass or high pass filters or we have resolution, resolution of signals and so on. So, these things are quite interesting to read in this area of application.