

Integral Transforms and Their Applications
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Lecture -48

Inverse Z – transform, Applications of Z – Transform - Part 3

$f(n) = \frac{1}{9} [10(-2)^n + 3n - 1] \quad \underline{\text{Ans}}$
 Eq 3. Fibonacci Sequence: $u_{n+1} = u_n + u_{n-1}$
 I.C's $u(0) = u(1) = 1$
 Sol'n Apply Z-transform to (1):
 $z[F(z) - f(0)] = F(z) + z^{-1}F(z) - u(-1)$
 Simplify (2)
 $\Rightarrow F(z) = \frac{z^2}{z^2 - z - 1}$
 $\Rightarrow u_n = z^{-1}[F] = z^{-1}\left[\frac{z^2}{z^2 - z - 1}\right]$
 $= z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] \quad \begin{matrix} \text{golden} \\ \text{ratio} \end{matrix}$
 $= z^{-1}\left[\frac{\frac{z^2}{a}}{(z-a)(z-b)}\right]$
 $= g_1 * g_2$
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$$f(n) = \frac{1}{9} [10(-2)^n + 3n - 1]$$

Example 3: Fibonacci Series:

$$u_{n+1} = u_n + u_{n-1}$$

$$\text{I.C's } u(0) = u(1) = 1$$

Solution: Apply z-transform:

$$\begin{aligned} z[F(z) - f(0)] &= \\ &= F(z) + z^{-1}F(z) - u(-1) \end{aligned}$$

Simplify 2:

$$\begin{aligned} F(z) &= \frac{z^2}{z^2 - z - 1} \\ u_n &= z^{-1}[F] = z^{-1} \left[\frac{z^2}{z^2 - z - 1} \right] \\ &= z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] \\ &= z^{-1} \left[\left(\frac{z}{z-a} \right) \left(\frac{z}{z-b} \right) \right] \\ &= g_1 * g_2 \end{aligned}$$

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$$\Rightarrow z^{-1} \left[\frac{z^2}{z^2 - z - 1} \right] = \frac{a^{n+1}}{(a-b)} \left[1 - \left(\frac{b}{a} \right)^{n+1} \right]$$

$$\Rightarrow u_n = \frac{a^{n+1} - b^{n+1}}{a-b} \quad n=0, 1, 2, \dots$$

Ex 4 Periodic Sol'n Find the IVP:

Sol'n Recall:

$$\begin{cases} u(n+2) - u(n+1) + u(n) = 0 \\ u(0) = 1 \\ u(1) = 2 \end{cases} \quad (1)$$

$$\begin{cases} z[u(n+2)] = z^2[F(z) - u(0)] \\ z[u(n+1)] = z[F(z) - 1] \\ z[u(n)] = z^2[F(z) - 2] \end{cases} \quad (2)$$

$$\Rightarrow u(n+2) - u(n+1) + u(n) = 0 \quad (3)$$

Using (2) in (1):

$$z^2[F(z) - z^2 - 2z] - \{z[F(z) - 1]\} + F(z) = 0$$

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$$z^{-1} \left[\frac{z^2}{z^2 - z - 1} \right] = \frac{a^{n+1}}{(a-b)} \left[1 - \left(\frac{b}{a} \right)^{n+1} \right]$$

$$u_n = \frac{a^{n+1} - b^{n+1}}{a-b}$$

Example 4: Periodic Solution:

Find the IVP:

$$u(n+2) - u(n+1) + u(n) = 0$$

Solution: Recall:

$$\begin{aligned} [z[u(n+2)]] &= z^2[F(z) - u(0)] \\ -zu(1) &= z^2(F(z) - 1) - 2z \\ z[u(n+1)] &= z[f(z) - 1] \\ z^2F(z) - z^2 - 2z \} &- zF(z) - z \} + F(z) = 0 \end{aligned}$$

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$$\Rightarrow F(z) = \frac{z^2 + 2}{z^2 - z + 1}$$

(Trick!) $\Rightarrow F(z) = \frac{z^2 - \frac{2}{z} + \frac{2}{z^2}z}{z^2 - z + 1}$

$$= \frac{z^2 - \frac{1}{2}z}{z^2 - z + 1} + \frac{\sqrt{3}(\frac{\sqrt{3}}{2})z}{z^2 - z + 1}$$

Take $\theta = \frac{\pi}{3}$ $= \frac{z^2 - \cos(\frac{\pi}{3})z}{z^2 - 2z \cos(\frac{\pi}{3}) + 1} + \frac{\sqrt{3} \sin(\frac{\pi}{3})z}{z^2 - 2z \sin(\frac{\pi}{3}) + 1}$

Recall! $Z[\cos n\theta] = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1} \quad (A)$

$Z[\sin n\theta] = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \quad (B)$

(A) $\equiv Z[\cos n\frac{\pi}{3}] \quad \rightarrow f(n) = \cos(n\frac{\pi}{3}) +$

(B) $\equiv \sqrt{3}Z[\sin n\frac{\pi}{3}] \quad \rightarrow \sqrt{3} \sin(n\frac{\pi}{3}) \quad \text{ETSC, IIT DELHI}$

$$\begin{aligned}
f(z) &= \frac{z^2 + z}{z^2 - z + 1} \\
F(z) &= \frac{z^2 - \frac{z}{2} + \frac{3}{2}z}{z^2 - z + 1} \\
&= \frac{z^2 - \frac{1}{2}z}{z^2 - z + 1} + \frac{\sqrt{3} \left(\frac{\sqrt{3}}{2} \right) z}{z^2 - z + 1} \\
&\text{Take } \theta = \frac{\pi}{3} \\
&= \frac{z^2 - \cos \frac{(\pi)}{3} z}{z^2 - 2z \cos \left(\frac{\pi}{2} \right) + 1} + \frac{\sqrt{3} \sin \left(\frac{\pi}{3} \right) z}{z^2 - 2z \sin \frac{\pi}{3} + 1}
\end{aligned}$$

Recall:

$$\begin{aligned}
z[\cos n\theta] &= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} \\
z[\sin n\theta] &= \frac{z\sin\theta}{z^2-2z\cos\theta+1}
\end{aligned}$$

$$\begin{aligned}
\mathbb{A} &\equiv z \left[\cos n \frac{\pi}{3} \right] \\
(\text{B}) &\equiv \sqrt{3} z \left[\sin n \frac{\pi}{3} \right]
\end{aligned}$$

$$\begin{aligned}
f(n) &= \cos(n \frac{\pi}{3}) + \\
&\quad \sqrt{3} \sin(n \frac{\pi}{3})
\end{aligned}$$

Theorem 1 (Summation of Series):

If $\sum_{k=1}^{\infty} f(k) = F(z)$, then:

- (1) $\sum_{k=1}^{\infty} f(k) = z^{-1} \left[\frac{z}{z-1} F(z) \right]$
- (2) $\sum_{k=1}^{\infty} f(k) = \lim_{z \rightarrow 1} F(z) = F(1)$

Proof: (1) L.H.S.: $g(n) = \sum_{k=1}^n f(k) = \underbrace{\sum_{k=1}^{n-1} f(k)}_{g(n-1)} + f(n)$

$$\Rightarrow g(n) = g(n-1) + f(n)$$

Apply Z-transform to (1):

$$\begin{aligned} \Rightarrow G(z) &= F(z) + z^{-1} G(z) \quad (g(0)=0) \\ \Rightarrow G(z) &= \frac{z}{z-1} F(z) \end{aligned}$$

Apply Inverse to (4): $g(n) = \sum_{k=1}^n f(k) = \mathcal{Z}^{-1} \left[\frac{z}{z-1} F(z) \right]$

Theorem 1: Summation of Series:

If $z[f(n)] = F(z)$, then:

1. $\sum_{k=1}^n f(k) = z^{-1} \left[\frac{z}{z-1} F(z) \right]$
2. $\sum_{k=1}^{\infty} f(k) = \lim_{z \rightarrow 1} F(z) = F(1)$

Proof: 1.

$$\begin{aligned}
 L'HS' : \quad g(n) &= \sum_{k=1}^n f(k) = \sum_{k=1}^n f(k) + f(n) \\
 g(n) &= g(n-1) + f(n) \\
 G(z) &= F(z) + z^{-1}G(z) \\
 G(z) &= \frac{z}{z-1}F(z) \\
 g(n) &= \sum_{k=1}^n f(k) = z^{-1} \left(\frac{z}{z-1}F(z) \right)
 \end{aligned}$$

The image shows a handwritten derivation on a grid background. It starts with the definition of $g(n) = \sum_{k=1}^n f(k)$ and relates it to the z-transform $Z\left[\frac{z}{z-1}F(z)\right]$. Then, it uses the final value theorem (FVT) to find the limit as $n \rightarrow \infty$ of $g(n)$. This limit is shown as $\lim_{z \rightarrow 1} z^{-1} \left(\frac{z}{z-1} F(z) \right)$, which simplifies to $\lim_{z \rightarrow 1} z F(z) = F(1)$. Below this, there is a note "Eq 1. Use z-trans. to show: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ ". A solution is provided: "Solt: $Z[x^n f(n)] = F(\frac{z}{x})$ (Recall)" and "Choose $f(n) = \frac{1}{n!} \Rightarrow Z[f] = \exp\left[\frac{z}{x}\right]$ ". The ETSC, IIT DELHI logo is visible in the top right corner.

$$g(n) = \sum_{k=1}^n f(k) = Z^{-1}\left[\frac{z}{z-1}F(z)\right]$$

Use FVT:

$$= \text{lt}_{z \rightarrow 1}(z-1)G(z)$$

$$\text{lt}_{z \rightarrow 1} zF(z) = F(1)$$

Example 1: Use z-transform to show:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Solution:

$$z[x^n f(n)] = F\left(\frac{z}{x}\right)$$

$$\text{Choose } f(n) = \frac{1}{n!}$$

$$z[f] = \exp\left[\frac{1}{z}\right]$$

$\Rightarrow z[x^n f(n)] = \exp\left[\frac{1}{z/x}\right] = \exp\left[\frac{x}{z}\right]$

$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \lim_{n \rightarrow \infty} \sum_{n=0}^k \frac{x^n}{n!} = \lim_{z \rightarrow 1} \underbrace{\exp\left[\frac{x}{z}\right]}_{FVT} = \exp(x)$

(Previous Thm 2nd part)

Eg 2 Find the sum $\sum_{n=0}^{\infty} a^n \sin nx$

Solⁿ: Sine transform: $z[f] = z[\sin nx]$

$= \frac{z \sin(x)}{z^2 - 2z \cos x + 1} \quad \text{①}$

To find $\sum_{n=0}^{\infty} a^n \sin nx = z[a^n \sin nx]$

$= F\left(\frac{z}{a}\right) = \frac{a z \sin x}{a^2 - 2a \cos x + 1} \quad \text{ETBC, IIT DELHI}$

$$z[x^n f(n)] = \exp\left[\frac{1}{z/x}\right] = \exp\left[\frac{x}{z}\right]$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{x^n}{n!} &= \lim_{n \rightarrow \infty} \sum_{n=0}^k \frac{x^n}{n!} \\ &\stackrel{lt_{z \rightarrow 1}}{=} \exp\left[\frac{x}{z}\right] \\ &= \exp(x) \end{aligned}$$

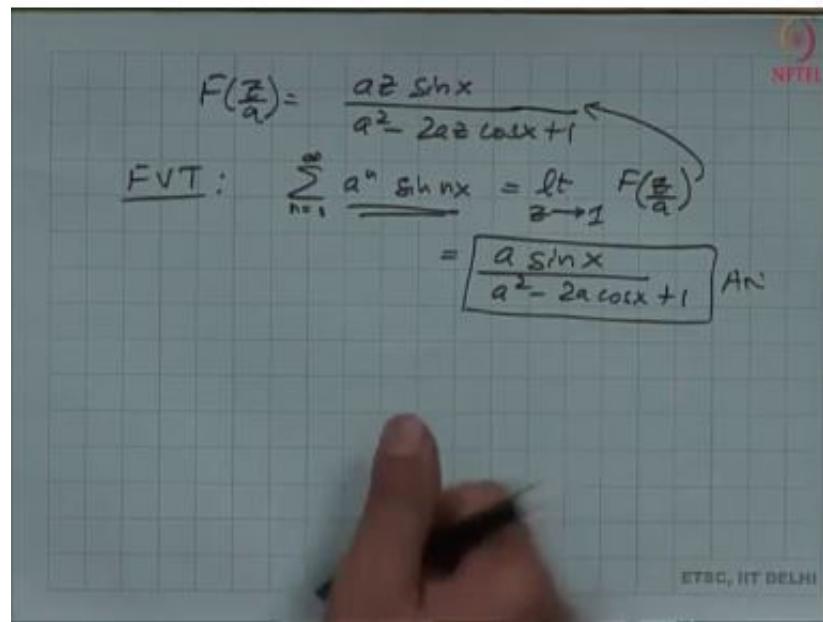
Example 2: Find the sum:

$$\sum_{n=0}^{\infty} a^n \sin nx$$

Solution: Sine transform

$$\begin{aligned} z[f] &= z[\sin nx] \\ &= \frac{z \sin(x)}{z^2 - 2z \cos x + 1} \end{aligned}$$

$$\begin{aligned} \sum_{n=0}^{\infty} a^n \sin nx &= z[a^n \sin nx] \\ &= F\left(\frac{z}{a}\right)^n = \frac{az \sin x}{a^2 - 2z} \end{aligned}$$



$$F\left(\frac{z}{a}\right) = \frac{az \sin x}{a^2 - 2az \cos x + 1}$$

$$\sum_{n=1}^{\infty} a^n \sin nx = \lim_{z \rightarrow 1} F\left(\frac{z}{a}\right)$$

$$\frac{a \sin x}{a^2 - 2a \cos x + 1}$$