Integral Transforms and Their Applications

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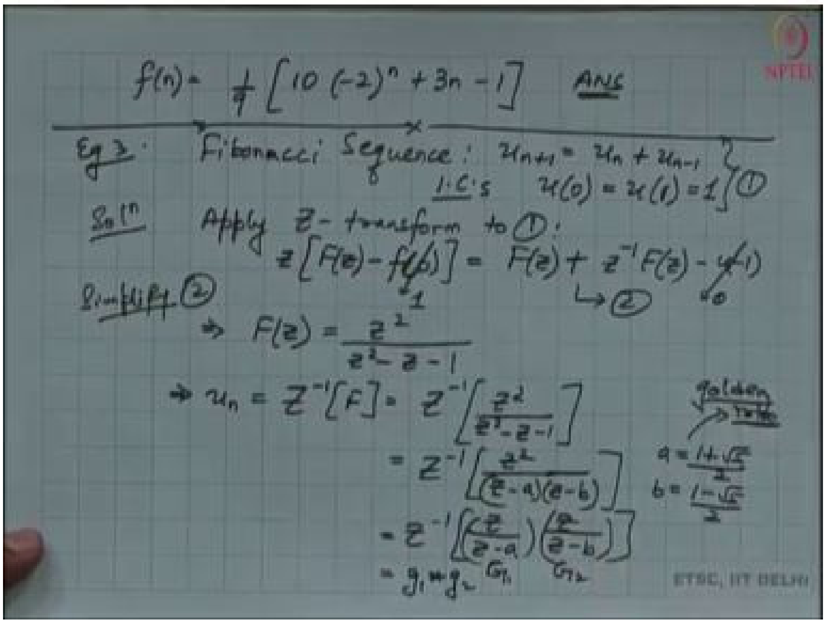
Lecture –48

Inverse Z – transform, Applications of Z – Transform - Part 3



$$f(n) = \frac{1}{9}\left[10(-2)^n + 3n - 1\right]$$

Example 3: Fibonacci Series:

$$u_{n+1} = u_n + u_{n-1}$$

$$\text{I } c\text{ 's} \quad u(0) = u(1) = 1$$

Solution: Apply z-transform:

$$z[F(z) - f(0)] =$$

$$= F(z) + z^{-1}F(z) - u(-1)$$

Simplify 2:

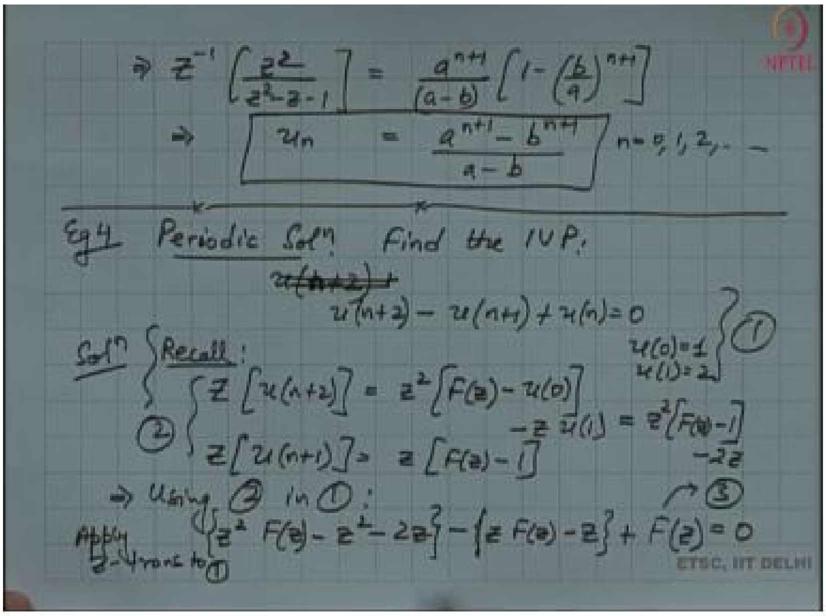
$$F(z) = \frac{z^2}{z^2 - z - 1}$$

$$u_n = z^{-1}[F] = z^{-1}\left[\frac{z^2}{z^2 - z - 1}\right]$$

$$= z^{-1}\left[\frac{z^2}{(z-\alpha)(z-b)}\right]$$

$$= z^{-1}\left[\left(\frac{z}{z-a}\right)\left(\frac{z}{z-b}\right)\right]$$

$$= g_1 * g_2$$



$$z^{-1}\left[\frac{z^2}{z^2-z-1}\right] = \frac{a^{n+1}}{(a-b)}\left[1-\left(\frac{b}{a}\right)^{n+1}\right]$$

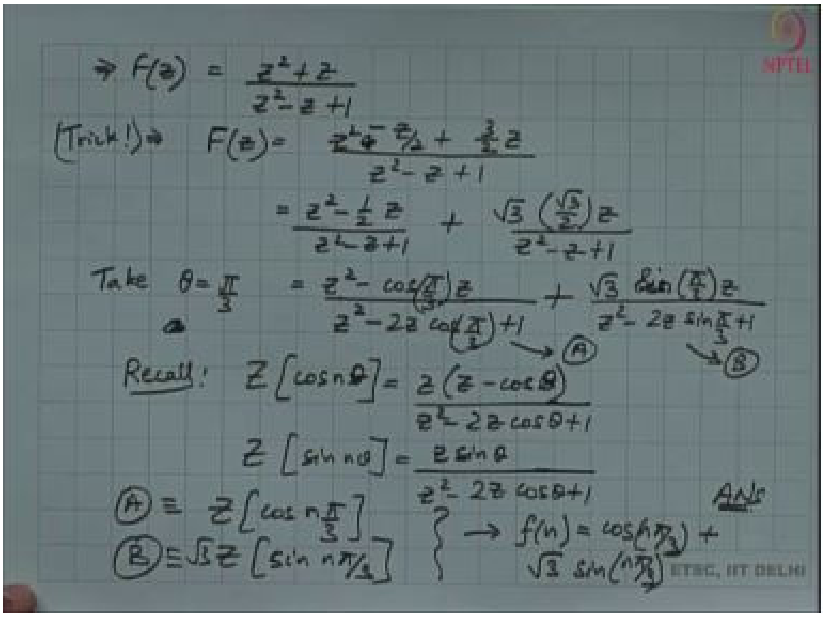
$$u_n = \frac{a^{n+1}-b^{n+1}}{a-b}$$

Example 4: Periodic Solution:
Find the IVP:

$$u(n+2)-u(n+1)+u(n)=0$$

Solution: Recall:

$$\big[z[u(n+2)] = z^2[F(z)-u(0)]$$
$$-zu(1) = z^2(F(z)-1)-2z$$
$$z[u(n+1)] = z[f(z)-1]$$
$$z^2F(z)-z^2-2z\big\} - zF(z)-z\big\} + F(z) = 0$$



$$f(z) = \frac{z^2 + z}{z^2 - z + 1}$$

$$F(z) = \frac{z^2 - \frac{z}{2} + \frac{3}{2}z}{z^2 - z + 1}$$

$$= \frac{z^2 - \frac{1}{2}z}{z^2 - z + 1} + \frac{\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)z}{z^2 - z + 1}$$

$$\text{Take } \theta = \frac{\pi}{3}$$

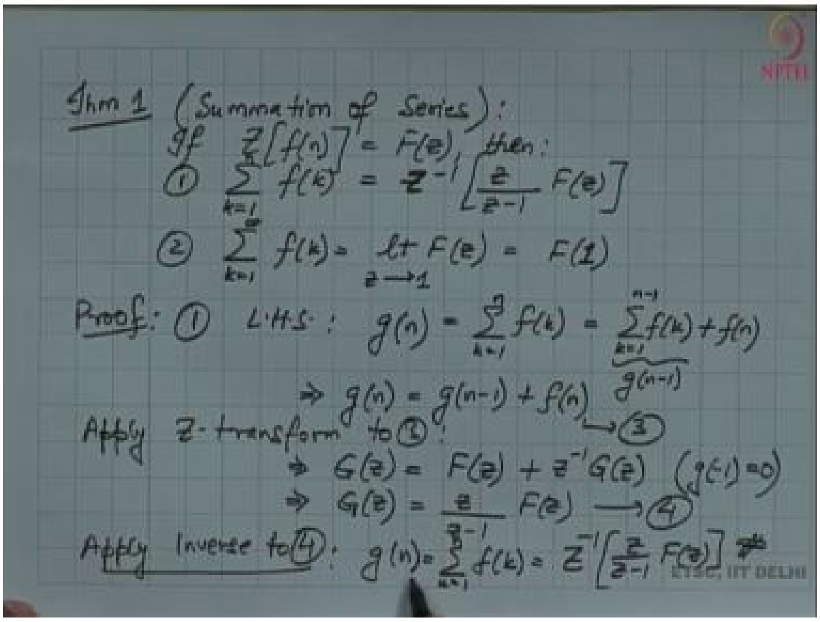
$$= \frac{z^2 - \cos\frac{(\pi)}{3}z}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1} + \frac{\sqrt{3}\sin\left(\frac{\pi}{3}\right)z}{z^2 - 2z\sin\frac{\pi}{3} + 1}$$

Recall:

$$z[\cos n\theta] = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$
$$z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\mathbb{A} \equiv z\left[\cos n\frac{\pi}{3}\right]$$
$$\text{(B)} \equiv \sqrt{3}z\left[\sin n\frac{\pi}{\pi/3}\right]$$

$$f(n) = \cos(n\tfrac{\pi}{3}) + \sqrt{3}\sin(n\tfrac{\pi}{3})$$



Theorem 1: Summation of Series:

$$\text{If } z[f(n)] = F(z) \text{ , then:}$$

$$1. \sum_{k=1}^{n} f(k) = z^{-1}\left[\frac{z}{z-1}F(z)\right]$$

$$2. \sum_{k=1}^{\infty} f(k) = \lim_{z\to1} F(z) = F(1)$$

Proof: 1.

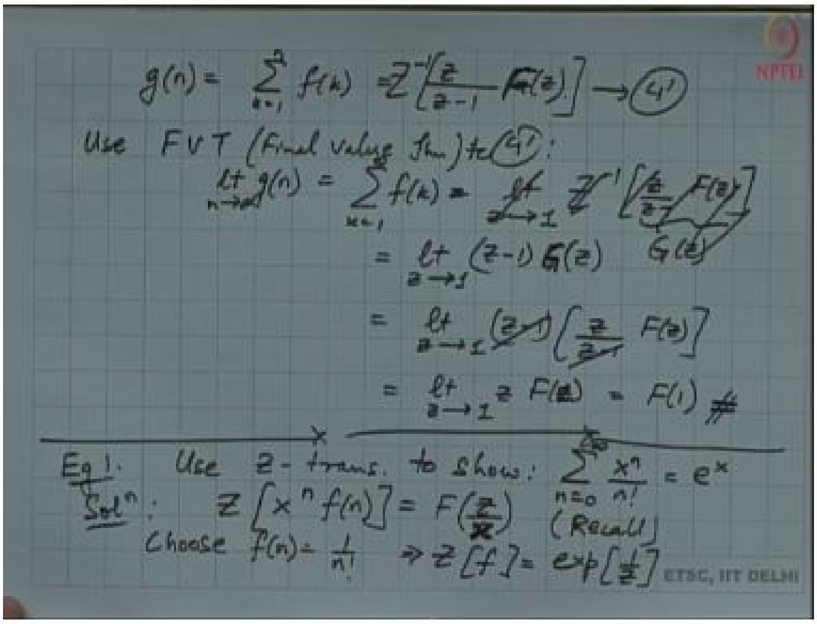
$$L'HS': \quad g(n) = \sum_{k=1}^{n} f(k) = \sum_{k=1}^{n} f(k) + f(n)$$

$$g(n) = g(n-1) + f(n)$$

$$G(z) = F(z) + z^{-1}G(z)$$

$$G(z) = \frac{z}{z-1}F(z)$$

$$g(n) = \sum_{k=1}^{n} f(k) = z^{-1}\left(\frac{z}{z-1}F(z)\right)$$



$$g(n) = \sum_{k=1}^{n} f(k) = Z^{-1}[\frac{z}{z-1}F(z)]$$

Use FVT:

$$= \mathrm{lt}_{z\to 1}(z-1)G(z)$$

$$\mathrm{lt}\ _{z\to 1} zF(z) = F(1)$$

Example 1: Use z-tyransform to show:

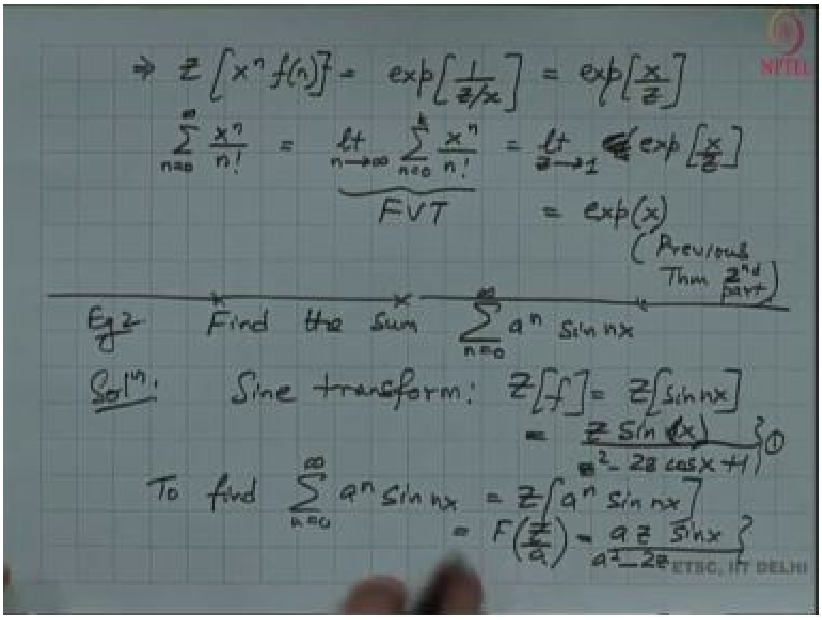
$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Solution:

$$z\left[x^n f(n)\right] = F\left(\frac{z}{x}\right)$$

$$\text{Choose } f(n) = \frac{1}{n!}$$

$$z[f] = \exp\left[\frac{1}{z}\right]$$



$$z\left[x^n f(n)\right] = \exp\left[\frac{1}{z/x}\right] = \exp\left[\frac{x}{z}\right]$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \mathrm{lt}_{n\to\infty} \sum_{n=0}^{k} \frac{x^n}{n!}$$

$$lt_{z\to 1} exp[\frac{x}{z}]$$

$$= \exp(x)$$

Example 2: Find the sum:

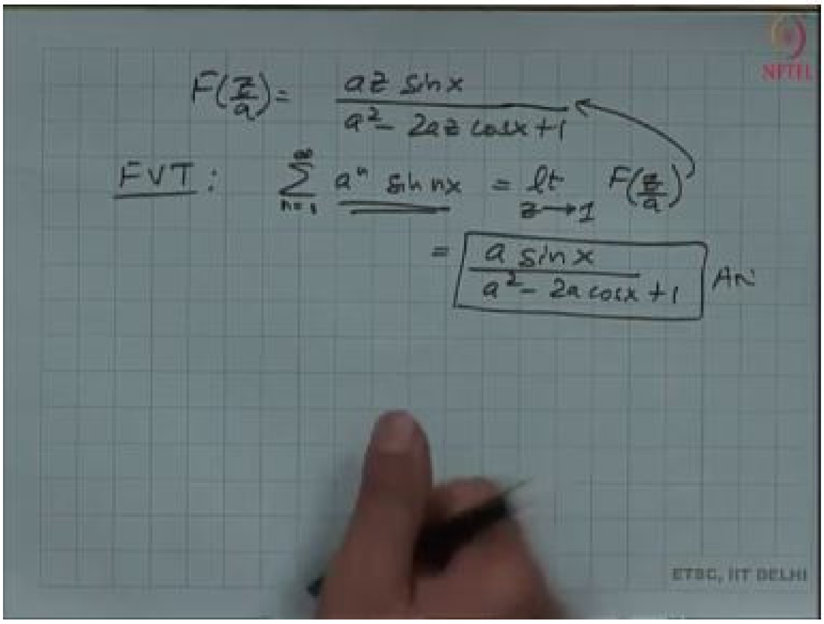
$$\sum_{n=0}^{\infty} a^n \sin nx$$

Solution: Sine transform

$$z[f] = z[\sin nx]$$
$$= \frac{z\sin(x)}{z^2 - 2z\cos x + 1}$$

$$\sum_{n=0}^{\infty} a^n \sin nx = z\left[a^n \sin nx\right]$$

$$= F\left(\frac{z}{a}\right)^n = \frac{azsinx}{a^2 - 2z}\Bigg\}$$



$$F\left(\frac{z}{a}\right) = \frac{az\sin x}{a^2 - 2az\cos x + 1}$$

$$\sum_{n=1}^{\infty} a^n \sin nx = \mathrm{lt}_{z\to 1} F\left(\frac{z}{a}\right)$$

$$\frac{a\sin x}{a^2 - 2a\cos x + 1}$$