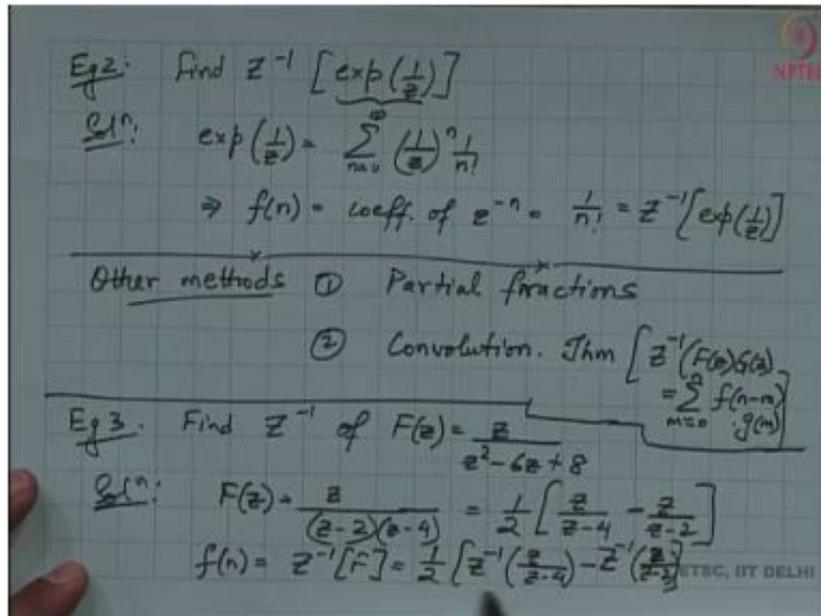


Integral Transforms and Their Applications
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 Lecture -47

Inverse Z - transform, Applications of Z - Transform Part 2



Example 2:

$$\text{Find } z^{-1} \left[\exp\left(\frac{1}{z}\right) \right]$$

Solution:

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \frac{1}{n!}$$

$$f(n) = \text{Coefficient of } z^{-n}$$

$$\frac{1}{n!} = z^{-1} \left[\exp\left(\frac{1}{z}\right) \right]$$

Other Methods:

1. Partial Functions
2. Convolution Theorem

Example 3:

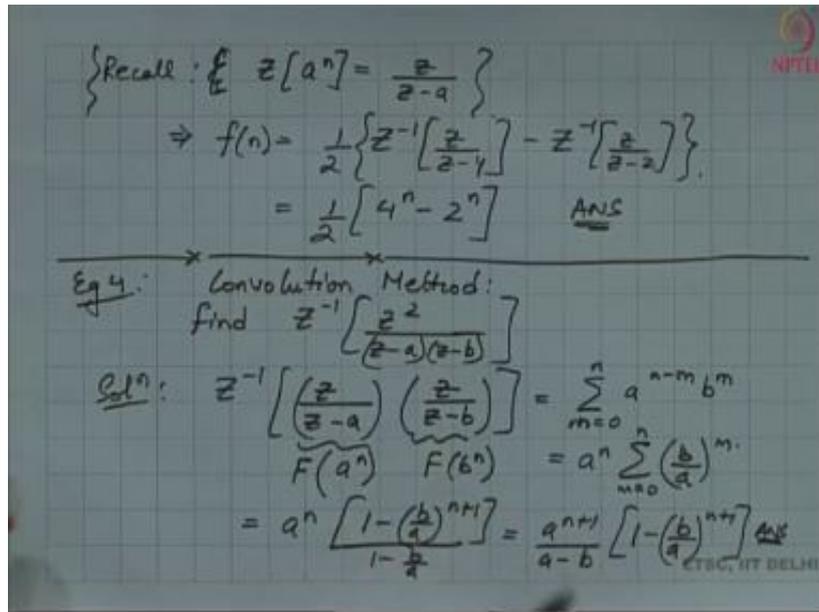
$$\text{Find } z^{-1} \text{ of } F(z) = \frac{z}{z^2 - 6z + 8}$$

Solution:

$$F(z) = \frac{z}{(z-2)(z-4)}$$

$$\frac{1}{2} \left[\frac{z}{z-4} - \frac{z}{z-2} \right]$$

$$f(n) = z^{-1}[F] = \frac{1}{2} \left[z^{-1} \left[\frac{z}{z-4} - \frac{z}{z-2} \right] \right]$$



Recall:

$$z[a^n] = \frac{z}{z-a}$$

$$f(n) = \frac{1}{2} \left\{ z^{-1} \left[\frac{z}{z-4} \right] - z^{-1} \left[\frac{z}{z-2} \right] \right\}$$

$$= \frac{1}{2} [4^n - 2^n]$$

Example 4: Convolution Theorem

$$\text{find } z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right]$$

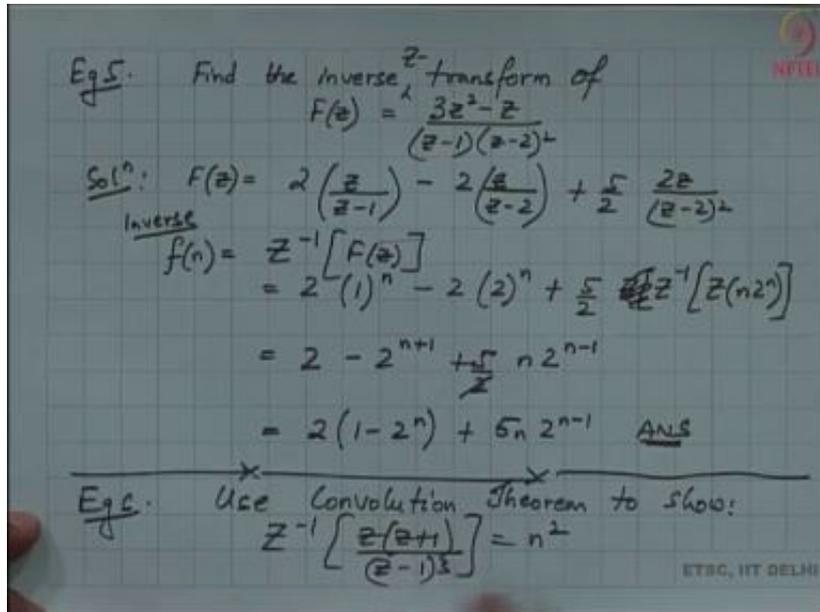
Solution:

$$z^{-1} \left[\left(\frac{z}{z-a} \right) \left(\frac{z}{z-b} \right) \right]$$

$$\sum_{m=0}^n a^{n-m} b^m$$

$$= a^n \sum_{m=0}^n \left(\frac{b}{a} \right)^m$$

$$= a^n \frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \frac{b}{a}} = \frac{a^{n+1}}{a-b} \left[1 - \left(\frac{b}{a}\right)^{n+1} \right]$$



Example 5:

Find the inverse z transform of

$$F(z) = \frac{3z^2 - z}{(z-1)(z-2)^2}$$

Solution:

$$\begin{aligned} F(z) &= 2 \left(\frac{z}{z-1} \right) - 2 \frac{z}{z-2} + \frac{5}{2} \frac{2z}{(z-2)^2} \\ f(n) &= z^{-1}[f(z)] \\ &= 2(1)^n - 2(2)^n + \frac{5}{2} z^{-1}[z(n2^n)] \\ &= 2 - 2^{n+1} + \frac{5}{2} n 2^{n-1} \\ &= 2(1 - 2^n) + 5n 2^{n-1} \end{aligned}$$

Example 6:

Use Convolution Theorem to show:

$$z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right] = n^2$$

$$\Rightarrow z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right] = z^{-1} \left[\frac{z}{(z-1)^2} \cdot \frac{z+1}{z-1} \right]$$

Choose $F(z) = \frac{z}{(z-1)^2}$; $G(z) = \frac{z+1}{z-1} = \frac{z}{z-1} + \frac{1}{z-1}$

$\left[z^{-1}[F(z)] = f(n) = n \cdot 1^n = n \right]$ $g(n) = z^{-1}[G(z)]$
Use 'series' method
 $= H(n) + H(n-1)$

$$z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right] = z^{-1} [F(z)G(z)]$$

$$= \sum_{m=0}^n f(n-m)g(m)$$

$$= \sum_{m=0}^n f(m)g(n-m)$$

$$= \sum_{m=0}^n m [H(n-m) + H(n-m-1)]$$

$$= \sum_{m=0}^n m H(n-m) + \sum_{m=0}^n m H(n-m-1)$$

$$\Rightarrow z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right]$$

$$z^{-1} \left[\frac{z}{(z-1)^2} \cdot \frac{z+1}{z-1} \right]$$

$$\text{Choose } F(z) = \frac{z}{(z-1)^2}$$

Use series method

$$G(z) = \frac{z+1}{z-1} = \frac{z}{z-1} + \frac{1}{z-1}$$

$$g(n) = z^{-1}[G(z)]$$

$$= H(n) + H(n-1)$$

$$z^{-1} \left[\frac{z(z+1)}{(z-1)^2} \right] = z^{-1}[F(z)G(z)]$$

$$= \sum_{m=0}^n f(n-m)g(m)$$

$$= \sum_{m=0}^n f(m)g(n-m)$$

$$= \sum_{m=0}^n m [H(n-m) + H(n-m-1)]$$

$$= \sum_{m=0}^n m H(n-m) + \sum_{m=0}^n m H(n-m-1)$$

$$s_1 = \sum_{m=0}^n m + \underbrace{H(n-m)}_{\substack{=1 \\ m \leq n}}$$

$$= \sum_{m=0}^n m = \frac{n(n+1)}{2}$$

$$s_2 = \sum_{m=0}^{n-1} m + \underbrace{H(n-m-1)}_{\substack{=1 \\ m \leq n-1}}$$

$$= \sum_{m=0}^{n-1} m = \frac{(n-1)n}{2}$$

ANS: $s_1 + s_2 = \frac{1}{2}n[n+1+n-1]$

$z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right] = n^2$

ANS

$$s_1 = \sum_{m=0}^n m + \underbrace{H(n-m)}$$

$$= \sum_{m=0}^n m$$

$$= \frac{n(n+1)}{2}$$

$$s_2 = \sum_{m=0}^{n-1} m + \underbrace{H(n-m-1)}$$

$$= \sum_{m=0}^{n-1} m = \frac{(n-1)n}{2}$$

$$s_1 + s_2 = \frac{1}{2}n[n+1+n-1]$$

$$\left[z^{-1} \left[\frac{z(z+1)}{(z-1)^3} \right] = n^2 \right]$$

Eg 7. $F(z) = \frac{z}{z-1}; |z| > 1$
 $G(z) = \frac{z}{z-1}; |z| < 1$

Solⁿ: $z^{-1}[F] = f(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{o.w. (n < 0)} \end{cases}$ (series-method)
 $z^{-1}(G) = g(n) = \begin{cases} 1 & n \leq 0 \\ 0 & n > 0 \end{cases}$ (Exercise)

Moral: z^{-1} is not unique

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Example 7:

$$F(z) = \frac{z}{z-1}; |z| > 1$$

$$G(z) = \frac{z}{z-1}; |z| < 1$$

Soln:

$$z^{-1}[F] = f(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{o.w. (n < 0)} \end{cases}$$

$$z^{-1}(G) = g(n) = \begin{cases} 1 & n \leq 0 \\ 0 & n > 0 \end{cases}$$

Moral: z^{-1} is not unique

Applications of z-transforms

Eg: 1st order difference Eqⁿ:
 Solve IVP: $f(n+1) - f(n) = 1; f(0) = 0$
 \downarrow (1)

Solⁿ: Apply z-transform to (1): $z[F(z) - f(0)] - F(z) = \frac{z}{z-1}$
 (Recalling z-transf. of $f(n+1)$) \rightarrow (2)

Simplify (2): $F(z)(z-1) = \frac{z}{z-1}$
 $\Rightarrow F(z) = \frac{z}{(z-1)^2} \rightarrow$ (3)

Take z^{-1} in (3) $\Rightarrow f(n) = z^{-1}\left[\frac{z}{(z-1)^2}\right] = n \cdot 1^n = \boxed{n}$ Ans

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Applications of z -transforms

1st order difference .

Solve (IVP , $f(n+1) - f(n) = 1; f(0) = 0$

Solution: Apply z -transform to 1:

$$= z[F(z) - f(0)]$$

$$-F(z) = \frac{z}{z-1}$$

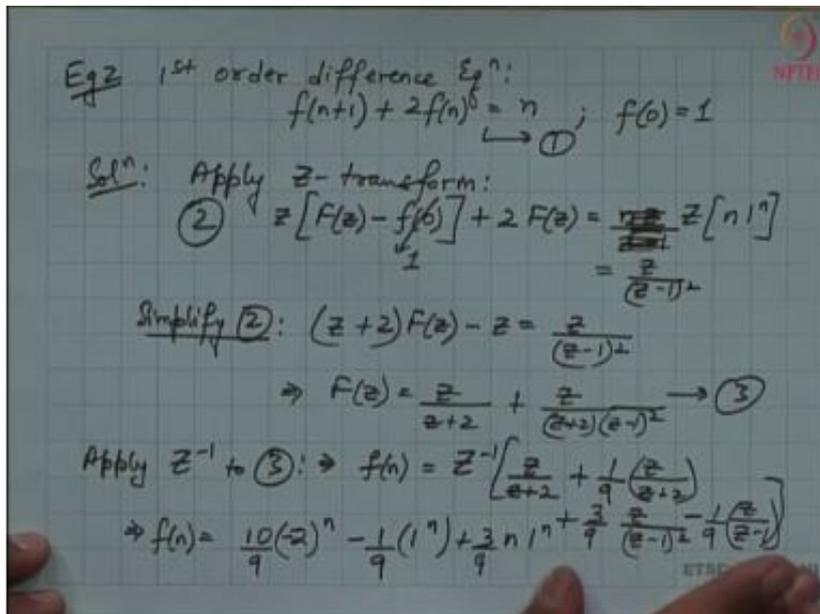
Simplify 2:

$$F(z) (z-1) = \frac{z}{z-1}$$

$$\Rightarrow F(z) = \frac{z}{(z-1)^2}$$

$$f(n) = z^{-1} \left(\frac{z}{(z-1)^2} \right)$$

$$= n1^n = n$$



Example 2: 1st order difference equation:

$$f(n+1) + 2f(n) = n; f(0) = 1$$

Solution: Apply z -transform

$$z[F(z) - f(0)] + 2F(z) =$$

$$z[n1^n]$$

Simplify 2:

$$(z+2)F(z) - z = \frac{z}{(z-1)^2}$$

$$\Rightarrow F(z) = \frac{z}{z+2} + \frac{z}{(z+2)(z-1)^2}$$

$$\begin{aligned} f(n) &= z^{-1} \left(\frac{z}{z+2} + \frac{1}{9} \left(\frac{z}{z+2} \right) \right. \\ &\quad \left. + \frac{3}{9} \frac{z}{(z-1)^2} - \frac{1}{9} \left(\frac{z}{z-1} \right) \right) \\ f(n) &= \frac{10(-2)^n}{9} - \frac{1}{9} (1^n) + \frac{3}{9} n 1^n \end{aligned}$$