Integral Transforms and Their Applications
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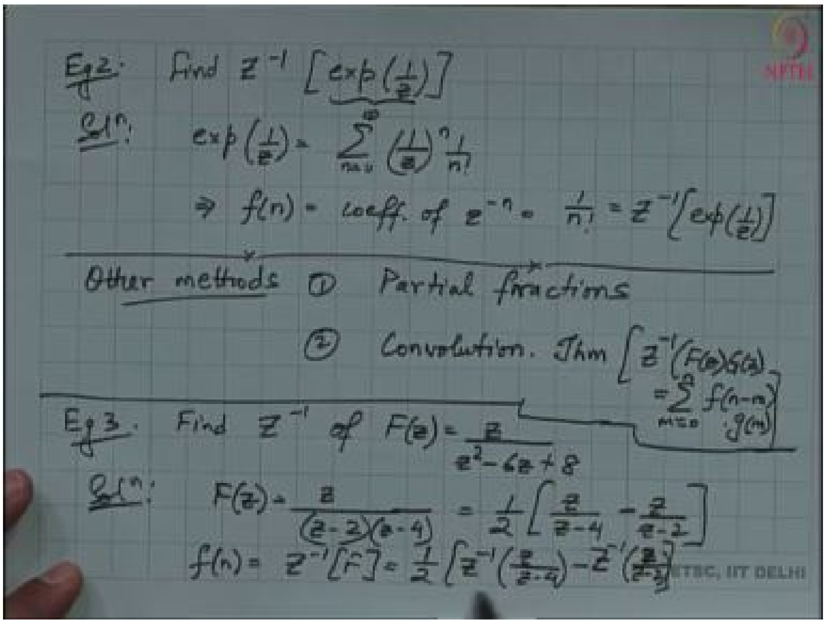
Lecture –47

Inverse Z - transform, Applications of Z - Transform Part 2



Example 2:

$$\text{Find } z^{-1}\left[\exp\left(\frac{1}{z}\right)\right]$$

Splotion:

$$\exp\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \frac{1}{n!}$$

$$f(n) = \text{Coefficient of } z^{-n}$$

$$\frac{1}{n!} = z^{-1}\left(\exp\left(\frac{1}{z}\right)\right]$$

Other Methods:

1. Partial Functions

2. Convolution Theorem

Example 3:

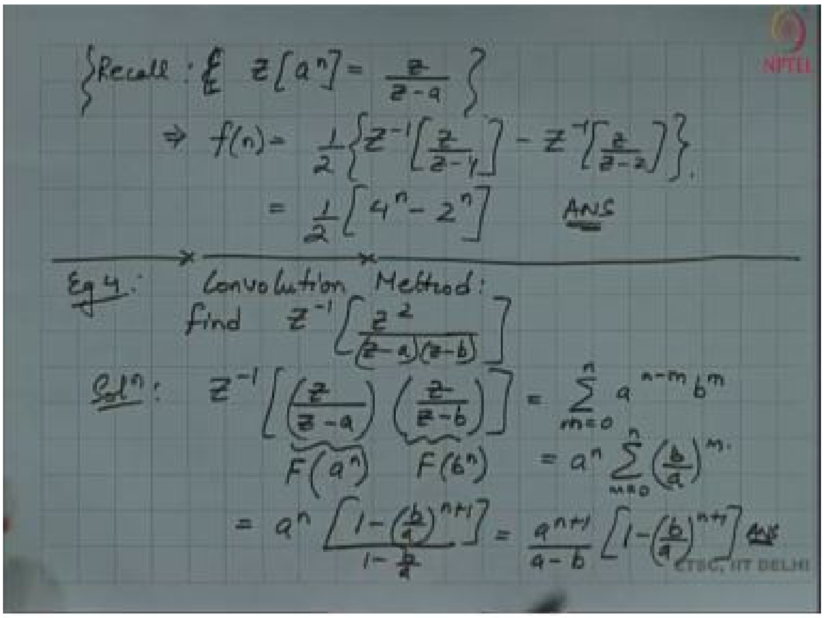
$$\text{Find } z^{-1} \text{ of } F(z) = \frac{z}{z^2 - 6z + 8}$$

Solution:

$$F(z) = \frac{z}{(z-2)(z-4)}$$

$$\frac{1}{2}\left[\frac{z}{z-4} - \frac{z}{z-2}\right]$$

$$f(n) = z^{-1}[F] = \frac{1}{2}\left[z^{-1}\left[\frac{z}{z-4} - \frac{z}{z-2}\right]\right]$$



Recall:

$$z\left[a^n\right] = \frac{z}{z-a}$$

$$f(n) = \frac{1}{2}\left\{z^{-1}\left[\frac{z}{z-4}\right] - z^{-1}\left[\frac{z}{z-2}\right]\right\}$$

$$= \frac{1}{2}\left[4^n - 2^n\right]$$

Example 4: Convolution Theorem

$$\text{find } z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$$

Solution:

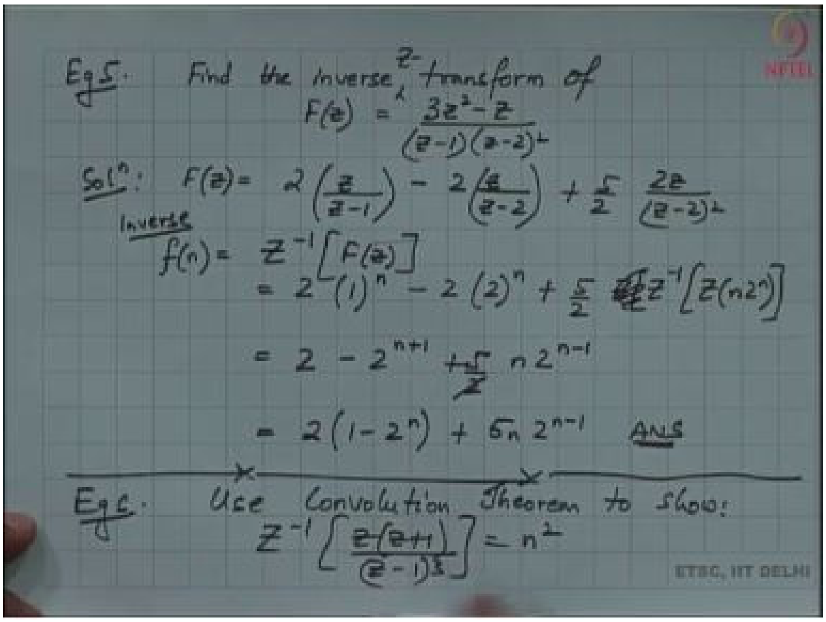
$$z^{-1}\left[\left(\frac{z}{z-a}\right)\left(\frac{z}{z-b}\right)\right]$$

$$\sum_{m=0}^{n} a^{n-m} b^m$$

$$= a^n \sum_{m=0}^{n} \left(\frac{b}{a}\right)^m$$

$$= a^n \frac{\left[1 - \left(\frac{b}{a}\right)^{n+1}\right]}{1 - \frac{b}{a}} =$$

$$\frac{a^{n+1}}{a-b}\left[1 - \left(\frac{b}{a}\right)^{n+1}\right]$$



Example 5:

Find the inverse z transform of
$F(z) = \frac{3z^2 - z}{(z-1)(z-2)^2}$

Solution:

$$F(z) = 2\left(\frac{z}{z-1}\right) - 2\frac{z}{(z-2)} + \frac{5}{2}\frac{2z}{(z-2)^2}$$

$$f(n) = z^{-1}[f(z)]$$

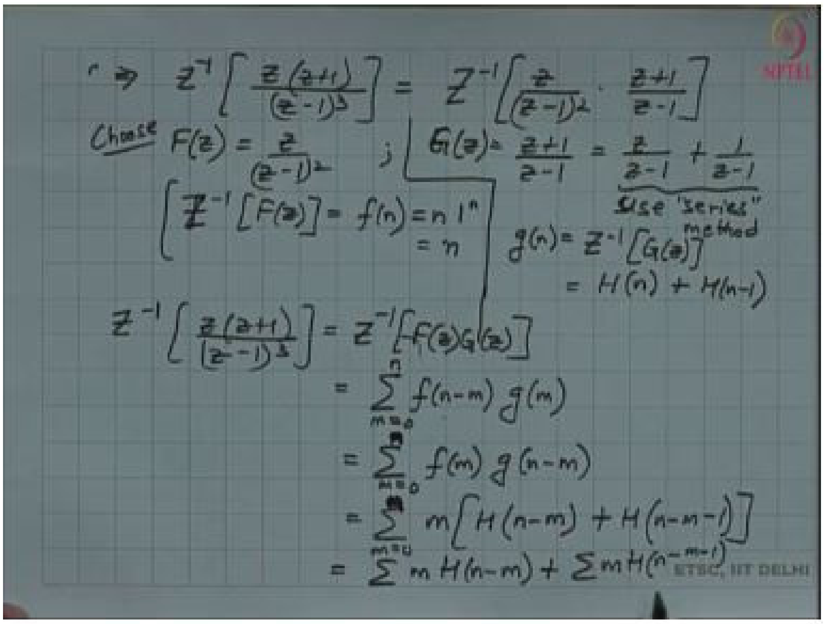
$$= 2(1)^n - 2(2)^n + \frac{5}{2}z^{-1}[z(n2^n)]$$

$$= 2 - 2^{n+1} + \frac{5}{2}n2^{n-1}$$

$$= 2\left(1 - 2^n\right) + 5n2^{n-1}$$

Example 6:

Use Convolution Theorem to show:
$z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right] = n^2$



$$\Rightarrow z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right]$$

$$z^{-1}\left[\frac{z}{(z-1)^2}\cdot\frac{z+1}{z-1}\right]$$

$$\text{Choose } F(z) = \frac{z}{(z-1)^2}$$

Use series method

$$G(z) = \frac{z+1}{z-1} = \frac{z}{z-1} + \frac{1}{z-1}$$

$$\begin{aligned} g(n) &= z^{-1}[G(z)] \\ &= H(n) + H(n-1) \end{aligned}$$

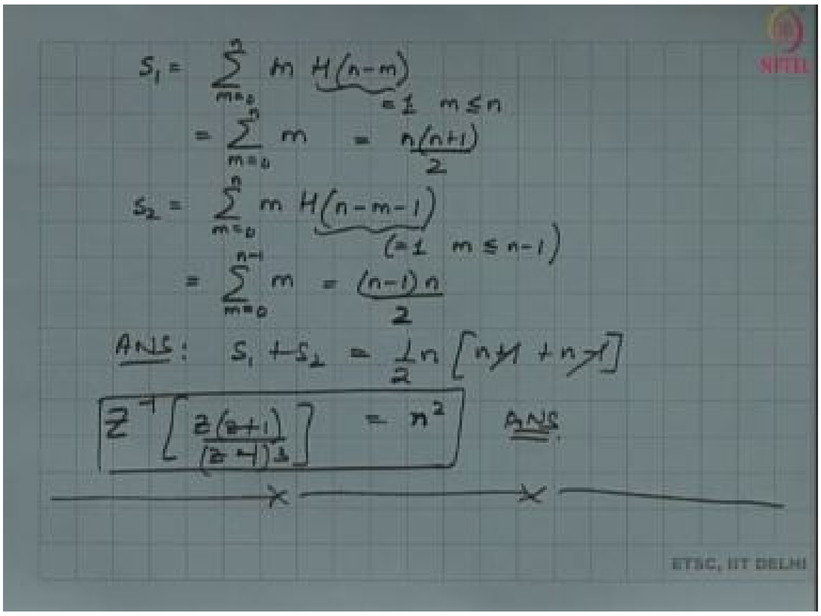
$$z^{-1}\left[\frac{z(z+1)}{(z-1)^2}\right] = z^{-1}[F(z)G(z)]$$

$$= \sum_{m=0}^{n} f(n-m)g(m)$$

$$= \sum_{m=0}^{n} f(m)g(n-m)$$

$$= \sum_{m=0}^{n} m[H(n-m) + H(n-m-1)]$$

$$= \sum^{m=0} mH(n-m) + \sum mH(n-m-1)$$



$$s_1 = \sum_{m=0}^{n} m + \underbrace{H(n-m)}$$

$$= \sum_{m=0}^{n} m$$

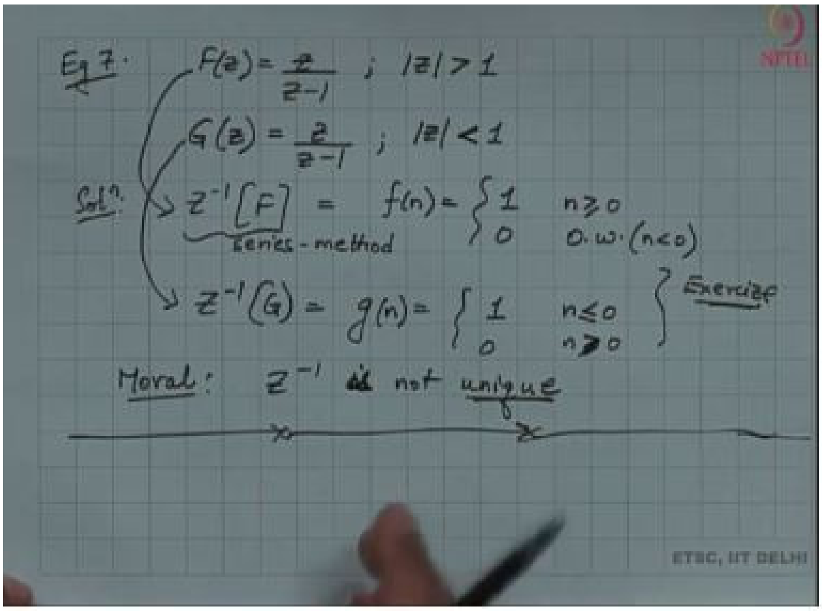
$$= \frac{n(n+1)}{2}$$

$$s_2 = \sum_{m=0}^{n} mH(n-m-1)$$

$$= \sum_{m=0}^{n-1} m = \frac{(n-1)n}{2}$$

$$s_1 + s_2 = \frac{1}{2} n[n+1+n-1]$$

$$\left[z^{-1}\left[\frac{z(z+1)}{(z-1)^3}\right] = n^2\right]$$



Example 7:

$$F(z) = \frac{z}{z-1}; |z| > 1$$

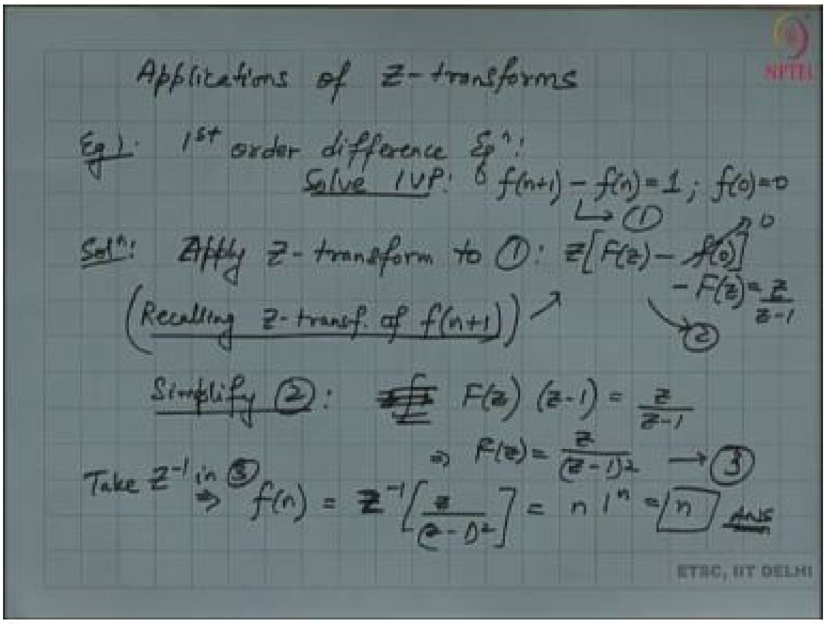
$$G(z) = \frac{z}{z-1}; |z| < 1$$

Soln:

$$z^{-1}[F] = f(n) = \left\{ \begin{array}{ll} 1 & n \geqslant 0 \\ 0 & 0.w(n < 0) \end{array} \right.$$

$$z^{-1}(G) = g(n) = \left\{ \begin{array}{ll} 1 & n \leqslant 0 \\ 0 & n > 0 \end{array} \right\}$$

Moral: z^{-1} is not unique



Applications of z -tranforms

1st order difference .

Solve $(IVP$, $f(n+1)-f(n)=1; f(0)=0$

Solution: Apply z-transform to 1:

$$=z[F(z)-f(0)]$$

$$-F(z)=\frac{z}{z-1}$$

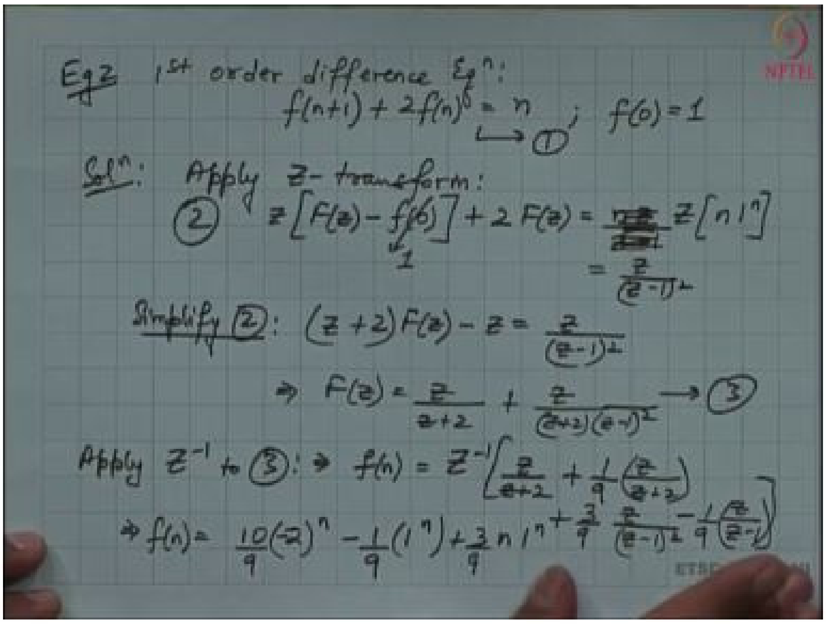
Simplify 2:

$$F(z) \quad (z-1)=\frac{z}{z-1}$$

$$\Rightarrow F(z)=\frac{z}{(z-1)^2}$$

$$f(n)=z^{-1}\left(\frac{z}{(z-1)^2}\right)$$

$$=n1^n=n$$



Example 2: 1st order difference equation:

$$f(n+1)+2f(n)=n; f(0)=1$$

Solution: Apply z-transform

$$z[F(z)-f(0)]+2F(z)=$$

$$z\left[n1^n\right]$$

Simplify 2:

$$(z+2)F(z)-z=\frac{z}{(z-1)^2}$$

$$\Rightarrow F(z)=\frac{z}{z+2}+\frac{z}{(z+2)(z-1)^2}$$

$$f(n) = z^{-1}\left(\frac{z}{z+2} + \frac{1}{9}\left(\frac{z}{z+2}\right)\right.$$

$$\left. + \frac{3}{9}\frac{z}{(z-1)^2} - \frac{1}{9}\left(\frac{z}{z-1}\right)\right)$$

$$f(n) = \frac{10(-2)^n}{9} - \frac{1}{9}\left(1^n\right) + \frac{3}{9}n1^n$$