Integral Transforms and Their Applications
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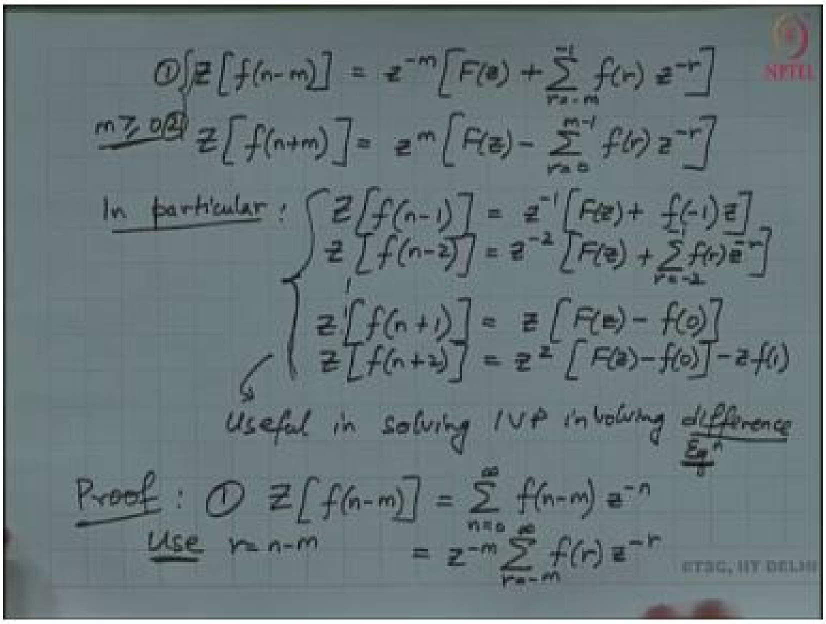
Lecture –45

Introduction to Z - transform Part 3



$$z[f(n-m)] = z^{-m}\left[F(z) + \sum_{r=-m}^{-1} f(r)z^{-r}\right].$$

$$z[f(n+m)] = z^{m}\left[F(z) - \sum_{r=0}^{m-1} f(r)z^{-r}\right], m >= 0$$

In particular:

$$z[f(n-1)] = z^{-1}[F(z) + f(-1)z]$$
$$z[f(n-2)] = z^{-2}\left[F(z) + \sum_{r=-2} f(n)e^{-r}\right]$$

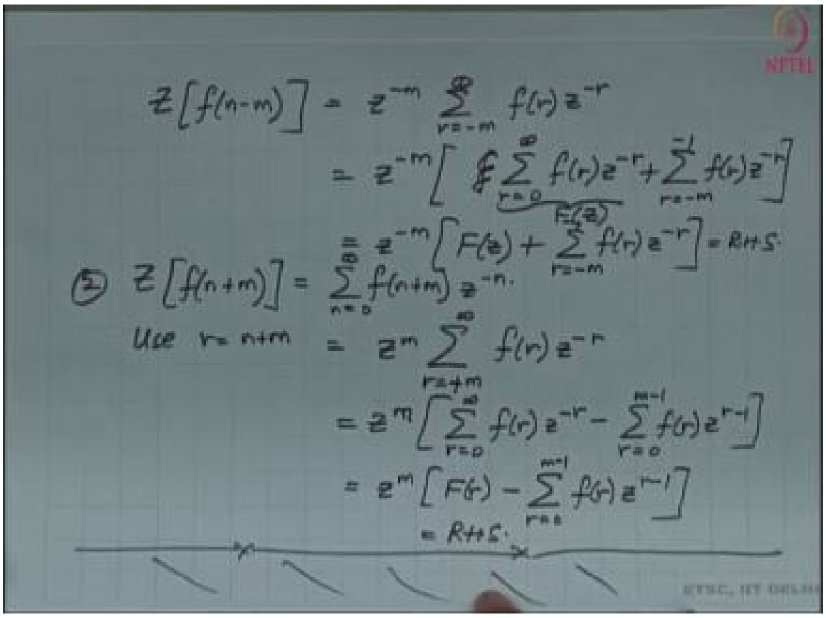
$$z[f(n+1)] = z \quad [F(z) - f(0)]$$
$$z[f(n+2)] = z^{2}[F(2) - f(0)] - zf(1)$$

, useful in solving IVP involving differnce equation

Proof:

$$(1)\ z[f(n-m)] = \sum_{n=0}^{\infty} f(n-m)z^{-n}$$

$$\text{use r=n-m} \quad = z^{-m}\sum_{r=-m}^{\infty} f(r)z^{-r}$$



$$z[f(n-m)] = z^{-m} \sum_{r=-m}^{\infty} f(r) z^{-r}$$

$$= z^{-m} \left[\sum_{r=0}^{\infty} f(r) z^{-r} + \sum_{r=-m}^{-1} f(r) z^{-r} \right]$$

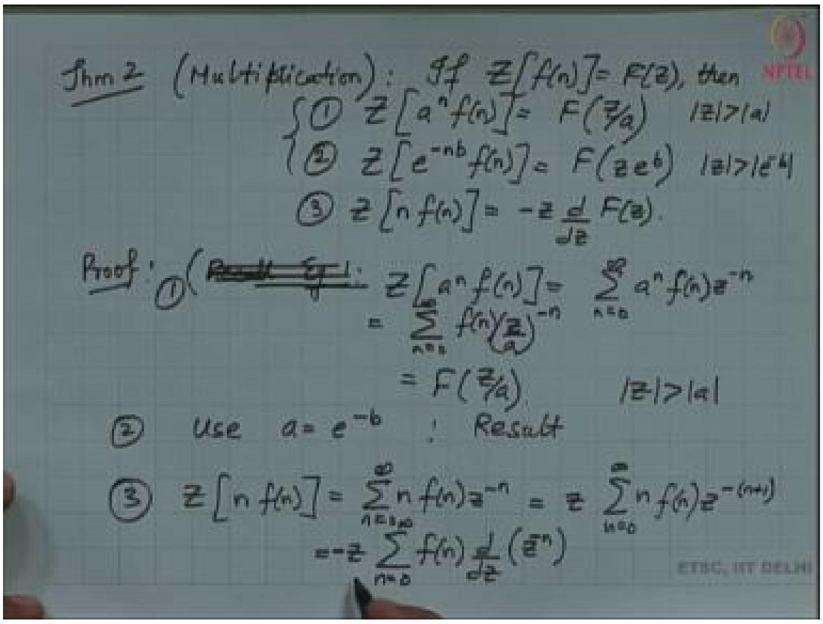
$$= z^{-m} \left[F(z) + \sum_{r=-m} f(r) z^{-r} \right] = RHS$$

$$(2) \;\; z[f(n+m)] = \sum_{n=0}^{\infty} f(n+m) z^{-n}$$

$$\text{use r=n+m } = z^{m} \sum_{r=+m}^{\infty} f(r) z^{-r}$$

$$= z^{m} \left[\sum_{r=0}^{\infty} f(r) z^{-r} - \sum_{r=0}^{m-1} f(r) z^{r-1} \right]$$

$$= z^{m} \left[F(r) - \sum_{r=0}^{m-1} f(r) z^{r-1} \right] = RHS$$



Theorem 2: Multiplication:

$$\text{If } z[f(n)] = f(z), \text{ then}$$

$$(1)\ z\left[a^n f(n)\right] = F(z/a) \quad |z| > |a|$$

$$(2)\ z\left[e^{-nb} f(n)\right] = F\left(ze^b\right) \quad |z| > \left|e^{-b}\right|$$

$$(3)\ z[nf(n)] = -z\frac{d}{dz}F(z)$$

Proof:

$$(1)\ z\left[a^n f(n)\right] = \sum_{n=0}^{\infty} a^n f(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n)\left(\frac{z}{a}\right)^{-n}$$

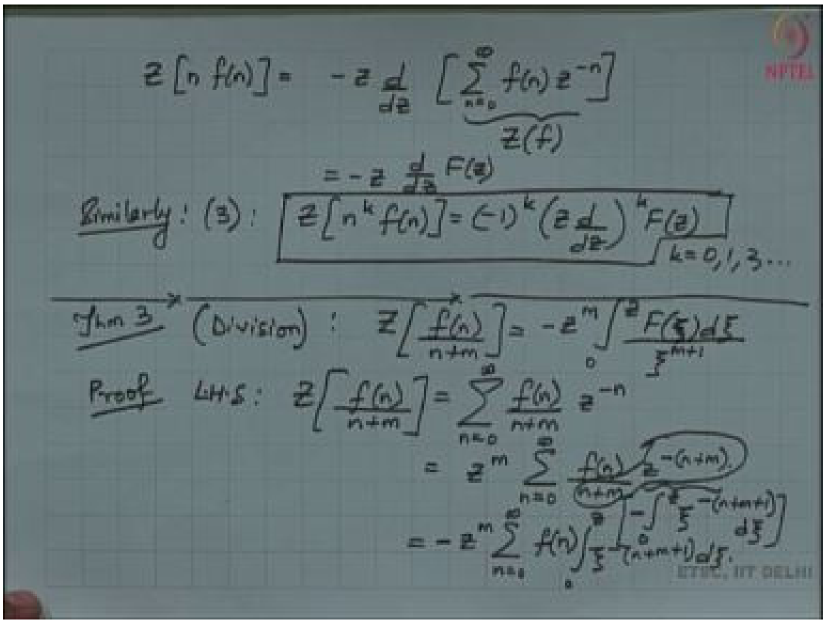
$$= F(z/a) \quad |z| > |a|$$

$$(2) \text{ use } a = e^{-b} : \text{ Result}$$

$$(3)\ z[nf(n)] = \sum_{n=0}^{\infty} nf(n) z^{-n}$$

$$= z\sum_{n=0}^{\infty} nf(n) z^{-(n+1)}$$

$$= -z\sum_{n=0}^{\infty} f(n)\frac{d}{dz}\left(rz^{-n}\right)$$



$$z[nf(n)] = -z\frac{d}{dz}\left[\sum_{n=0}^{\infty} f(n)z^{-n}\right]$$

$$= -z \quad \frac{d}{dz}F(z)$$

$$\text{similery: } (3):\left[z\left[n^k f(n)\right] = (-1)^k\left(z\frac{d}{dz}\right)^k F(z)\right], k = 0, 1, 2..$$

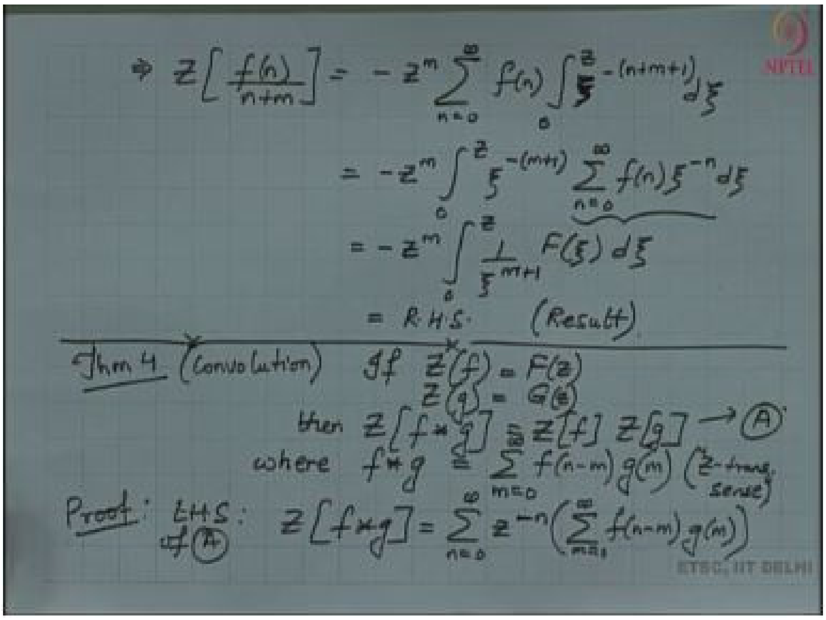
Theorem 3: Division:

$$Z\left[\frac{f(n)}{n+m}\right] = -z^m\int_0^2 \frac{F(\xi)d\xi}{\xi^{m+1}}$$

$$z[\frac{f(n)}{n+m}] = \sum_{n=0}^{\infty}\frac{f(n)}{n+m}z^{-n}$$

$$= z^m\sum_{n=0}^{\infty}\frac{f(n)}{n+m}z^{-(n+m)}$$

$$= -z^m\sum_{n=0}^{\infty} f(n)\int_0^z \xi^{-1}\left(n+m+1\right)d\xi$$



$$z\left[\frac{f(n)}{n+m}\right] = -z^m \sum_{n=0}^{\infty} f(n) \int_0^z \xi^{-(n+m+1)} d\xi$$

$$-z^m \int_0^z \xi^{-(m+1)} \sum_{n=0}^{\infty} f(n)\xi^{-n} d\xi$$

$$= -z^m \int_0^z \frac{1}{\xi^{m+1}} F(\xi) d\xi = RHS$$

Theorem 4: Convolution:

$$\text{If } z(f) = F(z), z(g) = G(z)$$

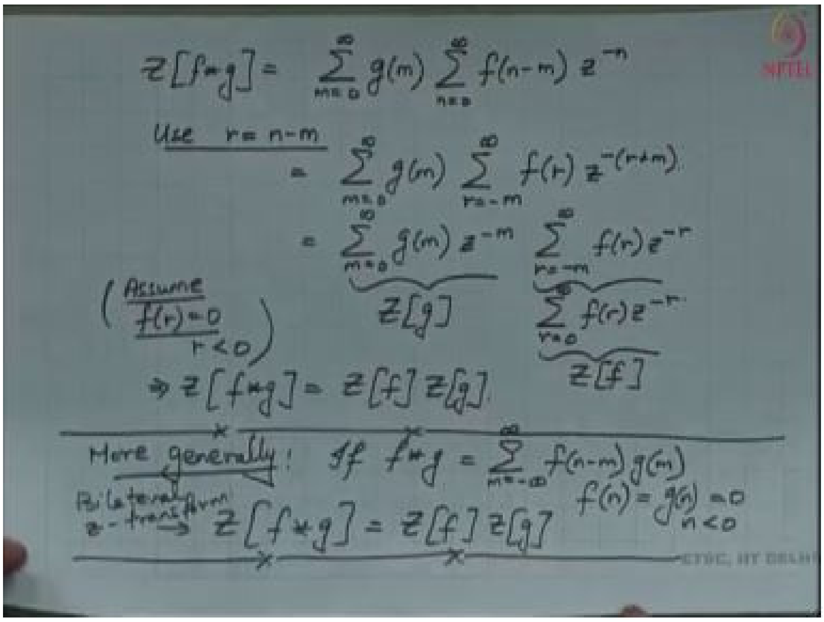
then:

$$z[f * g] = z[f]z[g]$$

$$\text{where } f * g = \sum_{m=0}^{\infty} f(n-m)g(m)$$

Proof:

$$z[f * y] = \sum_{n=0}^{\infty} z^{-n} \left(\sum_{m=0}^{\infty} f(n-m)g(n) \right)$$



$$z[f * g] = \sum_{m=0}^{\infty} g(m) \sum_{n=0}^{\infty} f(n-m) z^{-n}$$

$$\text{use r=n-m } = \sum_{m=0}^{\infty} g(m) \sum_{r=-m}^{\infty} f(r) z^{-(r+m)}$$

$$= \sum_{m=0}^{\infty} g(m) z^{-m} \sum_{r=-m}^{\infty} f(r) z^{-r}$$

Assume f(r)=0,$r < 0$

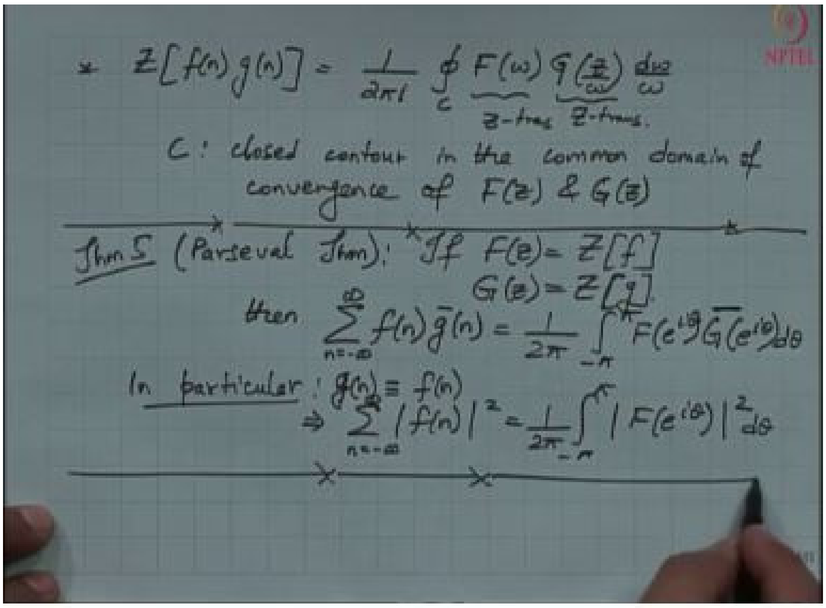
$$z[f * g] = z[f] z[g]$$

More generally,

$$\text{If } f * g = \sum_{m=-\infty}^{\infty} f(n-m) g(m)$$

Bilateral z-transform

$$z[f * g] = z[f] z[g], [f(n) = g(n) = 0, n < 0]$$



$$z[f(n)g(n)] = \frac{1}{2\pi i}\oint_{c} F(\omega)\left(\frac{z}{w}\right)\frac{dw}{w}$$

C: closed contout in the common domain of
convergence of $F(z)\&G(z)$

Theorem 5:
Parseval theorem:

$$\text{If } F(z) = Z[f]$$
$$G(z) = Z[g]$$

$$\text{then } \sum_{n=-\infty}^{\infty} f(n)\bar{g}(n) =$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F\left(e^{i\theta}\right)\bar{G}\left(e^{i\theta}\right)d\theta$$

In particular,

$$g(n) = f(n)$$
$$\textstyle\sum_{n=-\infty}^{\infty}\Big(f(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|F\left(e^{i\theta}\right)\right|^2 d\theta$$

thank you very much.