Integral Transforms and Their Applications
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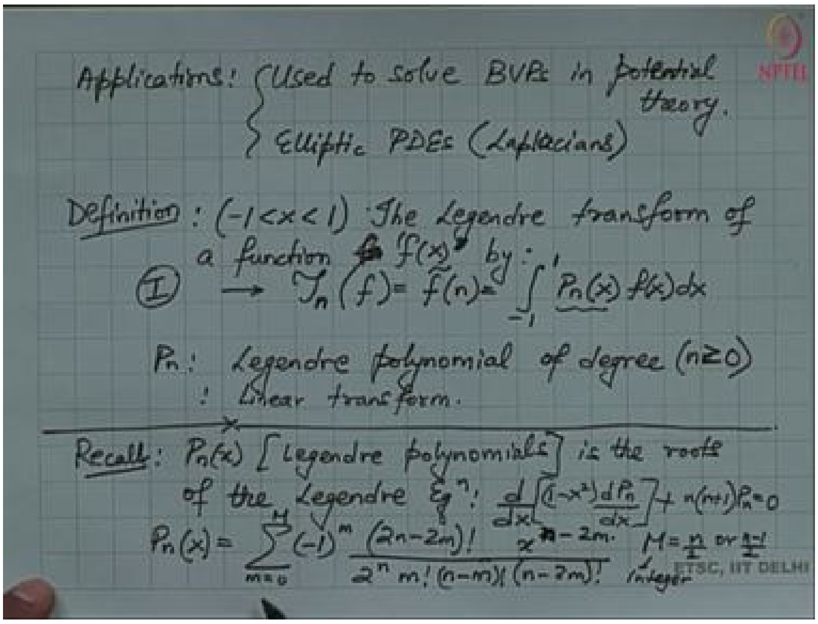
Lecture – 40
Introduction to Legendre Transform Part - 01

Good afternoon everyone. So, in today's lecture I am going to talk about another new transform known as the Legendre transform. As I have mentioned Legendre transforms are quite useful in solving problems in related to boundary value boundary value equations say specifically problems involving Laplacians or problems involving functions which has also called potential or potential functions. So, let us start the Legendre transforms the discussion on Legendre transforms.



So, as I said the applications of Legendre transforms are manifold, it is used to solve boundary value problems in potential theory and it is further quite useful in solving problems related to elliptic PDEs or PDEs for example, involving Laplacians ok. So, let me just give the definition of the Laplace transform first and then I am going to briefly recall some properties and the definition of Laplace sorry the Legendre transform. So, what I am given is, I am given a function and the function is defined in this range from $-1 to 1$, I define my Legendre transform I define my Legendre transform of a function of a function f of a function f(x) by the following integral:

So, I am going to define I am going to denote my Legendre transform with this

$$\mathcal{J}_n(f) = \tilde{f}(n) = \int_{-1}^{1} P_n(x) f(x) dx$$

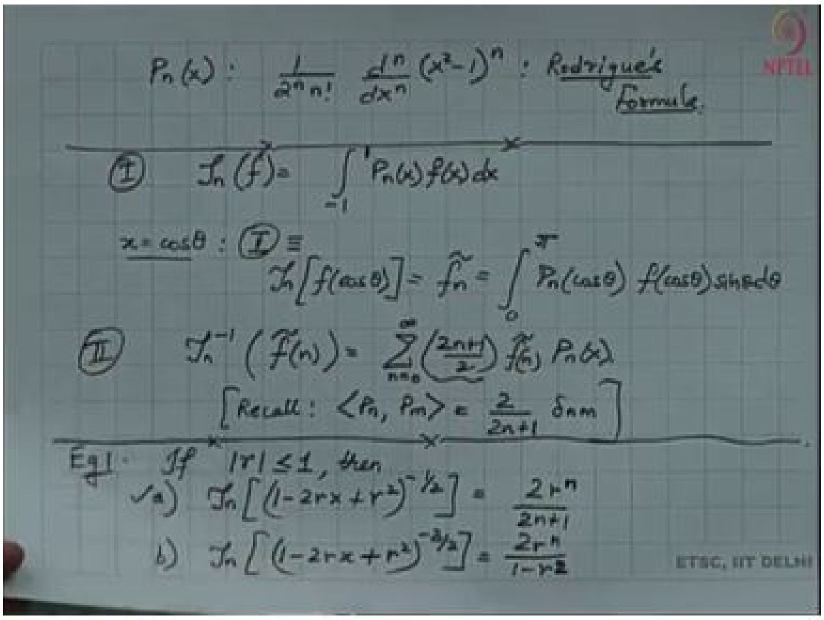
So, as I said P_n is the Legendre polynomial the Legendre polynomial of degree n ok. So, I am talking about integer order polynomials or integer order functions when I describe the Legendre transform. So, of course, this is a linear transformation, we are well aware of that.

So, before I move ahead and use show you how to use this transform let me just briefly recall the Legendre polynomial ok. So, I call this as my definition I and let us just briefly recall what do I mean by Legendre polynomials. So, Legendre polynomials denoted by $P_n(x)$.

So, this is also known as the Legendre polynomials, these are the roots of the Legendre equation given by

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dP_n}{dx}\right]+n(n+1)p_n=0$$

$$\mathrm{P_n}(x)=\sum_{m=0}^{M}(-1)^m\frac{(2n-2m)!x^{(n-2m)}}{2^n m!(n-m)!(n-2m)!}$$



So, then there is another formula for. So, what whatever the formula I have shown you for Legendre polynomials is rarely used. It is actually this particular formula that we use for deriving the legendary polynomial and the formula is given as follows.

$$P_n(x):\quad \frac{1}{2^n n!}\frac{d^n\left(x^2-1\right)^n}{dx^n}\ :\ \text{Redrigue's Formulae}$$

So, to derive the n th order polynomial this compact expression is used often known as the Rodriguez formula; Rodriguez formula ok.

Now coming back to my Legendre transform. So, what I have is, I have seen that according to my definition (I), my Legendre transform of a function is given by this integral :

$$(I.)\quad \mathcal{J}_n(f)=\int_{-1}^{1}P_n(x)f(x)dx$$

So, I am going to rewrite my Legendre transform in terms of the angle θ :

So, I is equal into Legendre transform of $f(cos\theta)$ and this is also equal to the following integral. Now instead of the original limits from $-1to+1$the new limit of θ, the θ varies from 0 to π.

$$\mathcal{J}_n[f(\cos\theta)] = \tilde{f}_n = \int_0^{\pi} P_n(\cos\theta) f(\cos\theta)\sin\theta d\theta$$

So, then of course, once I have described transform I also have the formula for the inverse transform. So, the inverse transform of the Legendre with respect to the Legendre polynomial is given by this following sum.

$$(II).\mathcal{J}_n^{-1}(\tilde{f}(n)) = \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)\tilde{f}(n)P_n(x)$$

We see that the inverse of the transform is given by the summation and this quantity in the front this particular factor comes out due to the orthogonality. So, recall. So, what I just said is the following. Recall that the Legendre polynomials are orthogonal and the orthogonality coefficient is given by

$$\text{Recall: } \langle P_n, P_m\rangle = \frac{2}{2n+1}\delta_{nm}$$

So, that is where this coefficient in this summation is coming from when we take the orthogonal the dot product of the transform. So, then, so, then let us let us look at one example a quick example ok. So, what I am given is the following. So, if I am given that a real number r is such that it lies between -1 to 1.

Example 1:If $|r| \leq 1$, then;

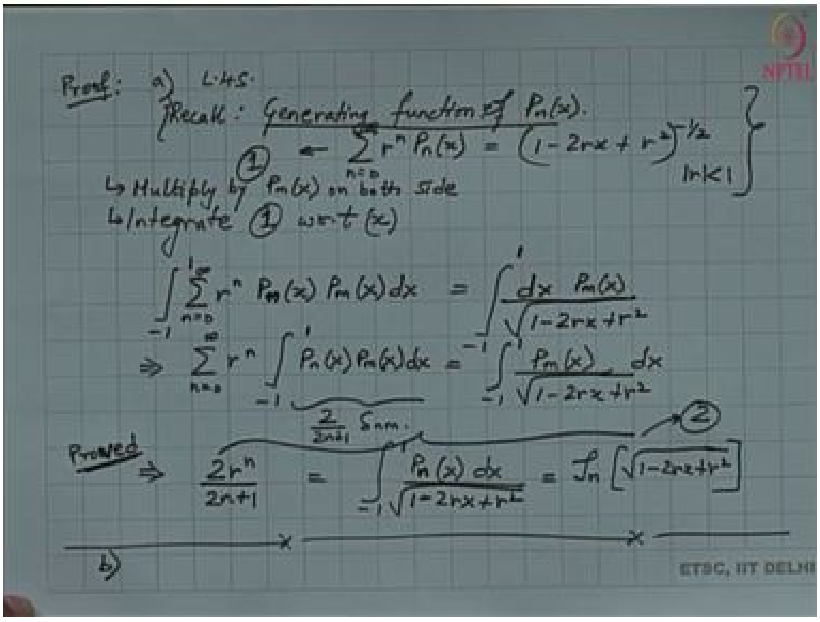
a)

$$\mathcal{J}_n\left[\left(1-2rx+r^2\right)^{-1/2}\right] = \frac{2r^n}{2n+1}$$

b)

$$\mathcal{J}_n\left[\left(1-2rx+r^2\right)^{-3/2}\right] = \frac{2r^n}{1-r^2}$$

So, let us look at this result. Let us start with the first one.



So, let us look at the left hand side of the first one. So, I am given that well. So, to begin prove the prove of this result, I have to first recall another result. I need to recall the generating function of the Legendre polynomial. So, I am recalling another property of Legendre polynomial namely the generating function. So, students should know that the generating function of Legendre polynomial is given by the following function.

$$\sum_{n=0}^{\infty} r^n P_n(x) = \left(1 - 2rx + r^2\right)^{\frac{-1}{2}} \text{......(1)}$$

And I have that r is for $r \leq 1$. Now I see that this particular x generating function is going to serve or purpose in solving in improving the result. If I were to integrate this expression this relation on both sides with respect to the limit from -1 to 1 and then we are going to get and then using the orthogonality condition, we are going to get a result.

So, I have this is a result of the Legendre generating function. So, integrate 1 with respect to x right. So, integrate 1 with respect to x and then what we need is, the first step is to multiply. So, I also need to use a orthogonality of the Legendre polynomials. So, multiply by $P_m(x)$ on both sides. So, that is the first step.

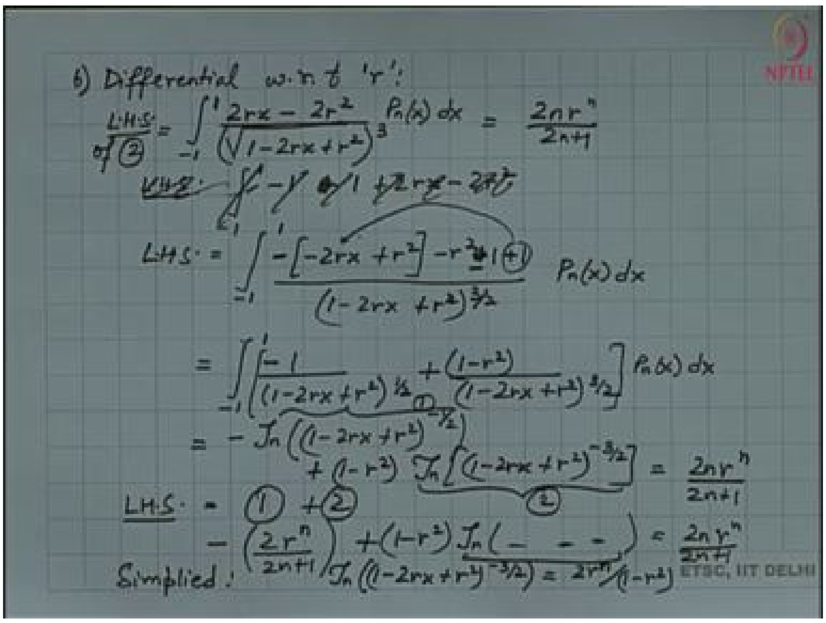
And then I need to integrate both sides with respect to x. So, let us see what happens.

$$\int_{-1}^{1} \sum_{n=0}^{\infty} r^n P_n(x)\, \mathrm{P_m}(x) dx = \int_{-1}^{1} \frac{dx P_m(x)}{\sqrt{1-2rx+r^2}}$$

$$\Rightarrow \sum_{n=0}^{\infty} r^n \int_{-1}^{1} P_n(x) P_m(x) dx = \int_{-1}^{1} \frac{P_m(x)}{\sqrt{1-2rx+r^2}} dx$$

$$\overset{\text{proved}}{\Rightarrow} \quad \frac{2r^n}{2n+1} = \int_{-1}^{1} \frac{P_n(x) dx}{\sqrt{1-2rx+r^2}} = \mathcal{J}_n[\sqrt{1-2rx+r^2}] \text{....(2)}$$

So moving on to show the next one, let us say that this is my this equality is my expression 2. So, now, I to prove the second part of the problem I am going to differentiate this whole expression this whole expression with respect to r.



b)So, what I am saying is differentiate in part b; if I were to differentiate with respect to r I get the following expression.

$$L \cdot H \cdot S \text{ of equation } 2 = \int_{-1}^{1} \frac{2rx - 2r^2}{\left(\sqrt{1-2rx+r^2}\right)^3} P_n(x)dx = \frac{2nr^n}{2n+1}$$

$$LHS = \int_{-1}^{1} \frac{-[-2rx+r^2] - r^2 - 1 + 1}{(1-2rx+r^2)^{3/2}} \mathrm{P_n}(x)dx$$

$$= \int_{-1}^{1} \left[\frac{-1}{(1-2rx+r^2)^{1/2}} + \frac{(1-r^2)}{(1-2rx+r^2)^{3/2}} \right] P_n(x)dx$$

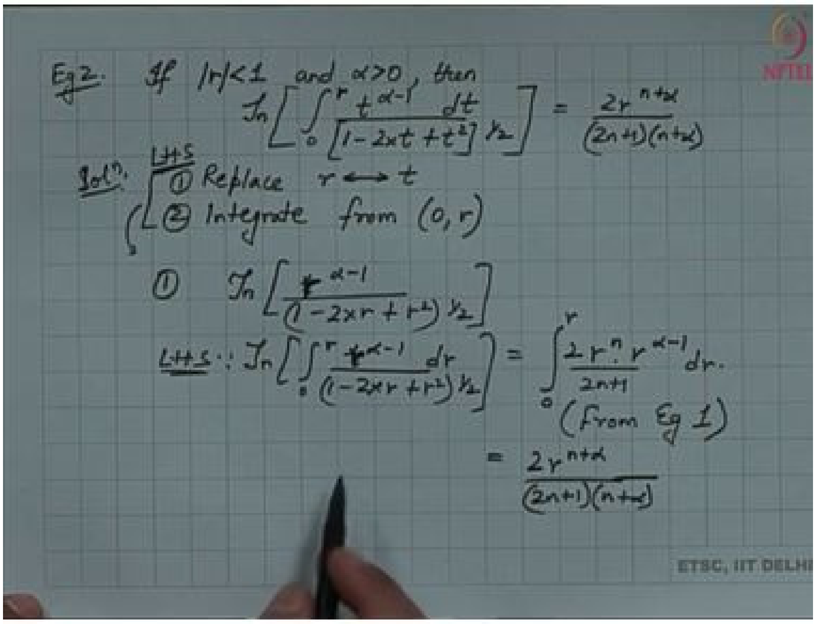
$$= -\mathcal{J}_n\left(\left(1-2rx+r^2\right)^{-1/2} \right) + (1-r^2)\mathcal{J}_n[(1-2rx+r^2)^{(-3/2)}] = \frac{2nr^n}{2n+1}$$

$$\text{LHS} = (1) + (2)$$

$$-\left(\frac{2r^n}{2n+1}\right) + \left(1-r^2\right)\mathcal{J}_n[(1-2rx+r^2)^{(-3/2)}] = \frac{2nr^n}{2n+1}$$

$$\text{Simplified} : \mathcal{J}_n[(1-2rx+r^2)^{(-3/2)}] = \frac{2nr^n}{1-r^2}$$

So, some of the steps I have skipped in between, but please simplify this expression and divide by $1 - r^2$ to come to this result.



Example 2:So, then let us look at some other examples. So, I have another example which says that if I am again I am given this parameter $r < 1$ and I am given $\alpha > 0$, then the Legendre transform the n^{th} order Legendre transform

$$\mathcal{J}_n\left[\int_0^r \frac{t^{\alpha-1}dt}{\sqrt{1-2xt+t^2]^{1/2}}}\right] = \frac{2r^{n+\alpha}}{(2n+1)(n+\alpha)}$$

Solution:Now, we see that this is a particular case of my previous example the first part. Namely, that I am now I have added another term in my integrating in my integrant, and I am also integrating with respect to r. So, then that gives the clue in order how to solve this. Namely I am going to replace my variable r with my variable t in. So, I am working I am working from the left hand side. So, I replace my variable r with t and I also integrate with respect to r from 0 to r. So, integrate from 0 to r:

$$\mathcal{J}_n\left[\frac{r^{\alpha-1}}{(1-2xr+r^2)^{1/2}}\right]...(1)$$

$$LHS:\mathcal{J}_n\left[\int_0^r\frac{r^{\alpha-1}dr}{(1-2xr+r^2)^{1/2}}\right]=\int_0^r\frac{2r^nr^{\alpha-1}}{2n+1}dr$$

Now, From Equation 1,

$$=\frac{2r^{n+\alpha}}{(2n+1)(n+\alpha)}$$