Integral Transforms and Their Applications
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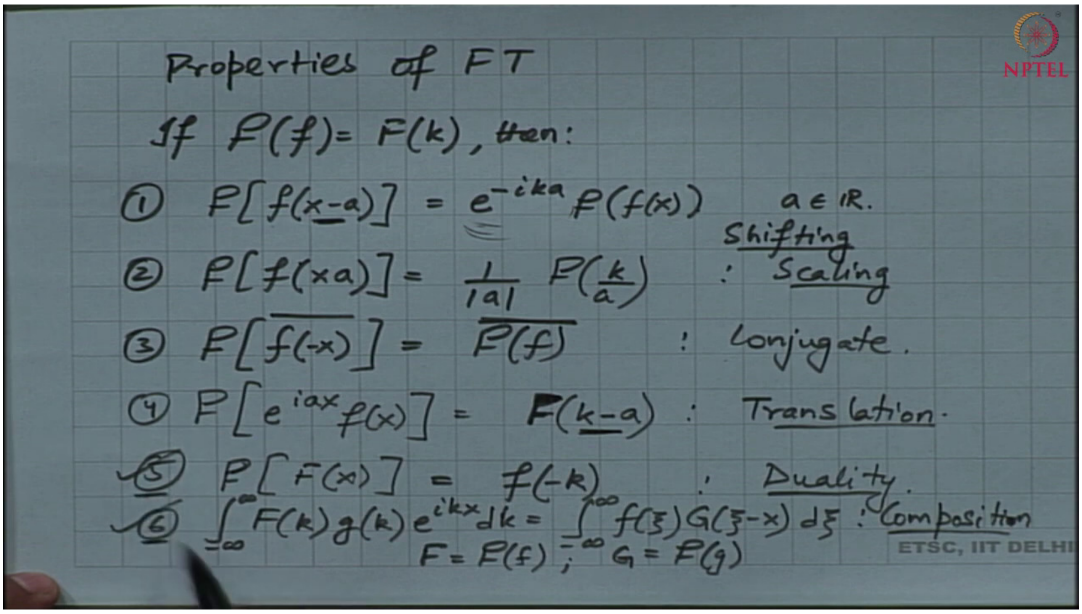
Lecture – 2
Introduction to Fourier Transforms Part - 01

So, good morning every one. So, in today's lecture I am going to talk about and continue on Fourier transforms and talk about the properties of Fourier transform followed by some important theorems, and important relations that are going to help us to figure out some Fourier transforms of more complicated functions. So, without waiting let us continue.



So, I will start today with the properties of Fourier transform. So, let me denote my Fourier transform of a function f as F(k). So, my k is the transformed variable and the transformed function for which I am evaluating the Fourier transform is small f. So, then some of the properties that are useful are I am going to prove and show some of these properties.

Properties of Fourier Transform:

1. Shifting property: Notice that if I shift my argument of the function by small a. Then that is corresponding to multiplying with this factor e to the power negative i k a in the transformed plane.

$$F[f(x-a)] = e^{-ika}F(f(x)) \quad a \in \mathbb{R}$$

where, well a is any real number

2. Scaling property: which says that f(x) times a is 1 by the absolute value of a, the Fourier transform of k by a right. So, I call this as my scaling property.

$$F[f(xa)] = \frac{1}{|a|}F\left(\frac{k}{a}\right)$$

You can see that if I were to scale my argument of the function by a then that is corresponding to this factor outside the transformed value.

3. Conjugate Property: Then another property is that if I have to evaluate the Fourier transform of the conjugate of f.

$$F[\overline{f(-x)}] = \overline{F(f)}$$

So, Fourier transform of the conjugate of $f(-x)$ is also equal to the Fourier transform of f whole conjugate right, I called this as the conjugate property.

4. Translation Property:

$$F\left[e^{iax}f(x)\right] = F(k-a)$$

Notice that this translation property is the shifting in the transformed plane while in the shifting property we have a shifting in the actual or the physical domain right, and notice how is factors are multiplied with the corresponding sign.

5. Duality Property: Then we have another property called the duality property; it tells us that the Fourier transformed of capital F(x) notice. This capital F is the transform function with the variable x replaced by k replace with x. So, Fourier transform of the Fourier transform function is the actual function evaluated at this variable small k.

$$f[F(x)] = f(-k)$$

So, I call this as the duality property

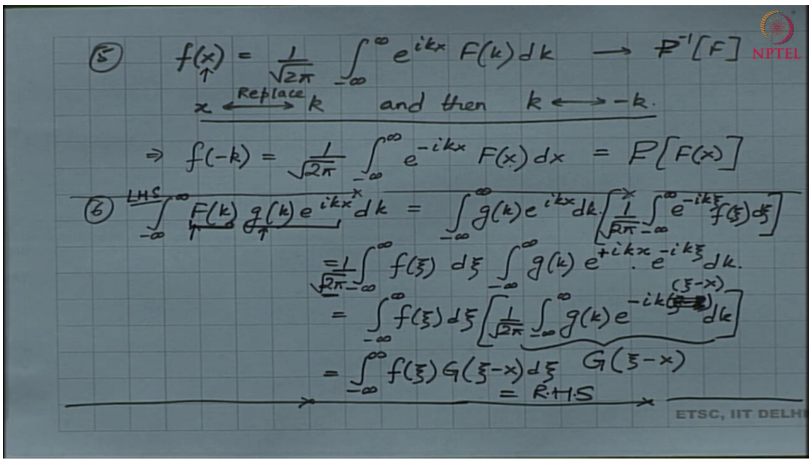
6. Composition Property:

$$\int_{-\infty}^{\infty} F(k)g(k)e^{ikx}dk = \int_{-\infty}^{+\infty} f(\xi)G(\xi - x)d\xi$$

where,

$$F = F(f), G = F(g)$$

Now most almost all this relations could be derived very nicely by using the definition of Fourier transform that I described in the last lecture. The only point of confusion could be possibly in this relation, this relation and this relation and this is what I am going to derive it today. The rest I leave it to the students as an exercise to see that these relations are certainly true. So, I am going to derive these two today. So, let us look at the duality property.



So, the 5th one; so, by the definition of integral transforms sorry the Fourier transforms I have that f is defined to be the inverse of capital F.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(k) dk$$

So, notice that this is nothing, but the inverse the inverse transform of capital F right. So,now I am going to replace x by k; I am going to replace x by k and k by x everywhere. And then I can replace k by minus k. So, what happens is that when I do that. So, instead of x I get a k here with the minus sign. So, I get,

$$f(-k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} F(x) dx = F[F(x)]$$

then you can easily see that this definition is nothing, but the definition of Fourier transform of capital F of x right. So, the derivation is quite straightforward if we use these change of variables ok. Then let us look at the next one the composition. So, I have the left hand side of the composition LHS is given by,

$$\int_{-\infty}^{\infty} F(k) g(k) e^{ikx} dk$$

but the variable is the Fourier variable or the spectral variable. So, let me just you know take this variable separately and let me introduce the definition of Fourier transform for capital F. So, I get,

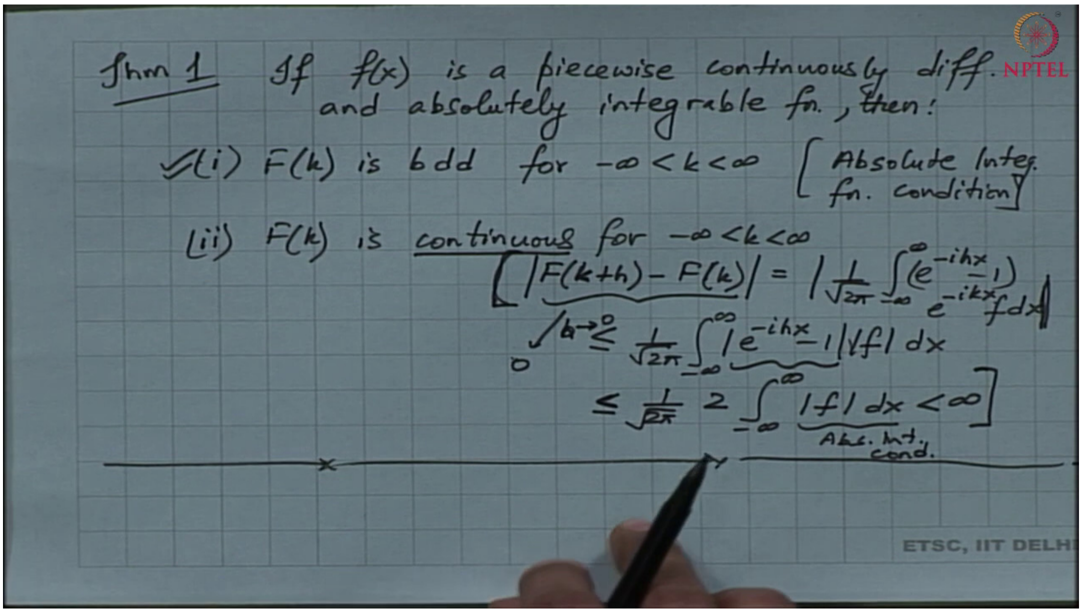
$$= \int_{-\infty}^{\infty} g(k) e^{ikx} dk \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ik\xi} f(\xi) d\xi$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) d\xi \int_{-\infty}^{\infty} g(k) e^{+ikx} e^{-ik\xi} dk$$

$$= \int_{-\infty}^{\infty} f(\xi) d\xi \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{-ik(\xi - x)} dk \right].$$

$$= \int_{-\infty}^{\infty} f(\xi) G(\xi - x) d\xi$$

and hence that is your right hand side. So, that completes my proof for the 6th case, the 6th result ok. Similarly other results can be proved. Now let me just highlight few results in the name of theorems which we will be using. So, I am not going to show I am not going to prove most of the theorems, but I will prove those theorems for which will be required for us to use in the letter application. So, to begin with I am going to talk about let me say that this is my theorem 1. So, the theorem 1 says that if f is a piecewise; f is a piecewise continuously differentiable function; continuously differentiable and absolutely integrable function then I have the following:



1. F(k) is bounded for all values of this transformed variable k

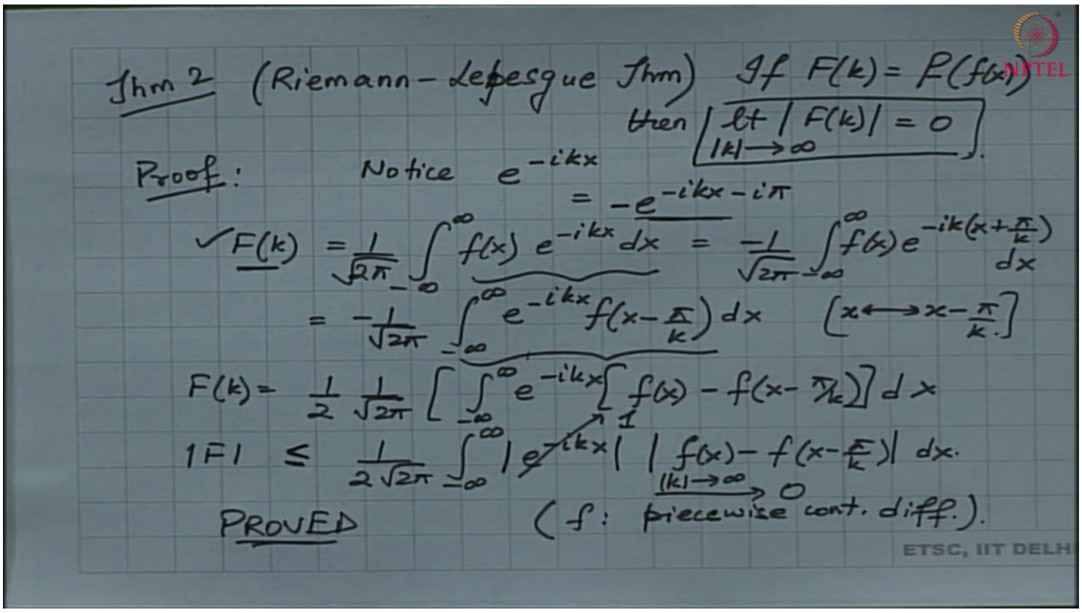
$$-\infty < k < \infty.$$

Now this is quite state forward without showing much proof this is quite state forward because I know that the Fourier transform is only defined for absolutely integrable function right. So, for absolutely integrable function condition this result will follow ok. So, that can be shown right away. The second result that I have is that if F k is a Fourier transform then F of k is continuous for all values of k right. So, in particular it tells; so, I am just going to show you in just few lines how can we show that.

$$|F(k+h) - F(k)| = |\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(e^{-ihx} - 1\right) |. \leqslant \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left|e^{-ihx} - 1\right| f|dx$$

$$\leq \frac{1}{\sqrt{2\pi}} 2 \int_{-\infty}^{\infty} |f|dx < \infty|$$

So, of course, the inequality remains I get 1 by square root 2 pi times 2 integral from minus infinity to infinity mod f dx and I know that this is finite because of the absolutely integrable condition; so, absolutely integrable condition right. Now in particular so that completes the proof that F of x, F of k is continuous ok. Well there was one more issue that this further this thing, this thing goes to 0 as h goes to 0. So, you can check that this particular factor this goes to 0 as h goes to 0. So, that will also be needed to show the continuity of Fof k. So, here I have just shown you in some brief steps on related to the proof of this part 2. So, then I am going to introduced one more lemma or one more result.



I am going to talk about well let me tell this as a theorem. So, the theorem is a very famous lemma called the Riemann - Lebesgue lemma or Riemann Lebesgue theorem. So, let me call this as a theorem because this will be applicable to our case of Fourier transforms. So, it tells me that if I have the Fourier transform of a function defined by

Theorem 2:

$$F(k) = F(f(x)) then, \begin{array}{c} \text{lt } |F(k)| = 0 \\ |k| \to \infty \end{array}$$

Now, to show the proof this is quite a quick proof so, let me show that.

$$e^{-ikx} = -e^{-ikx-i\pi}$$

So, this is also equals e to the power negative i k x negative i pi and I put a minus sign you can expand this e to the power i pi and you will see that that is equal to negative 1 and that cancels with this negative 1 ok.

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$= \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ik\left(x+\frac{\pi}{k}\right)} dx$$

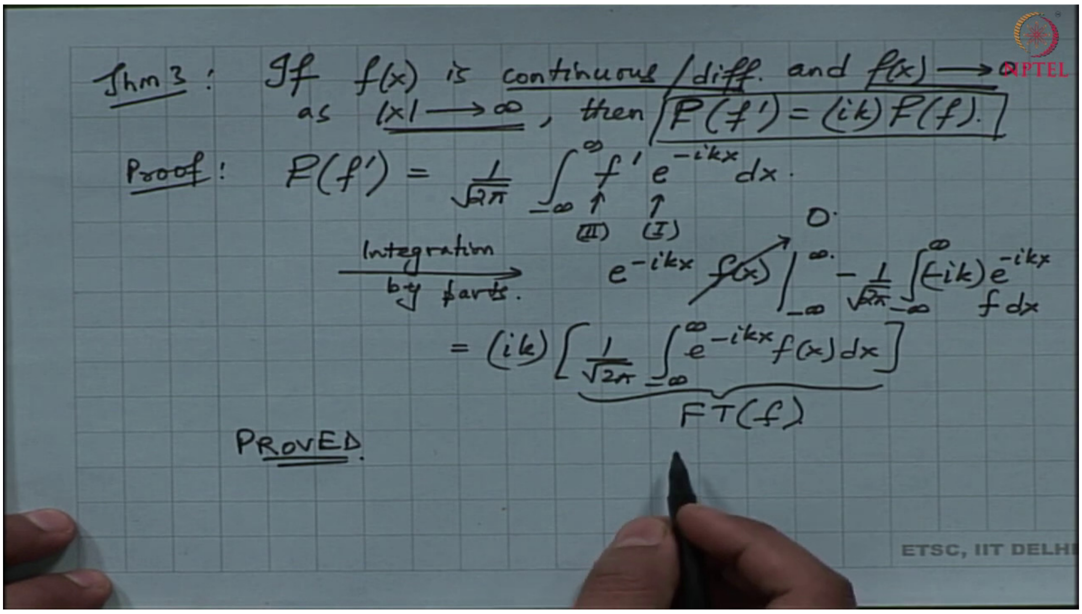
notice that if I do a change of variables, I change my variable I replace x by x minus pi by k. So, I do a change of variables

$$= -\frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} e^{-ikx} f\left(x - \frac{\pi}{k}\right) dx$$

Now you see that the Fourier transform of this function is this thing that is underlined and it is also equal to this thing which is underlined. So, which means I can defined by Fourier transform to be half of these two value. So, all I have done is I have added these two relations and taken a half because they are both equal.

$$F(k) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\left[\int_0^{\infty} e^{-ikx}[f(x) - f(x - \frac{\pi}{k})]dx\right].$$

So, notice that if we take the absolute value then of course, the absolute value is bounded above by the following integral which is the integral of the function in absolute terms And you can see that this goes to 0 in the limit k goes to infinity right because, we have that f is piecewise continuously differentiable. So, that is why this limit inside this integral goes to 0 as we have this wave number going to infinity and that proves are result which is the following right ok. So, then so, these results will be quite useful later on when we do some problems.



Theorem 3: So, then there is another result which I denoted as a theorem it talks about the derivative of Fourier transform. So, if I have a function f which is a continuous function. In fact, continuously differentiable and I have f goes to 0 in the limit x goes to infinity or minus infinity then

$$F\left(f'\right) = (ik)F(f)$$

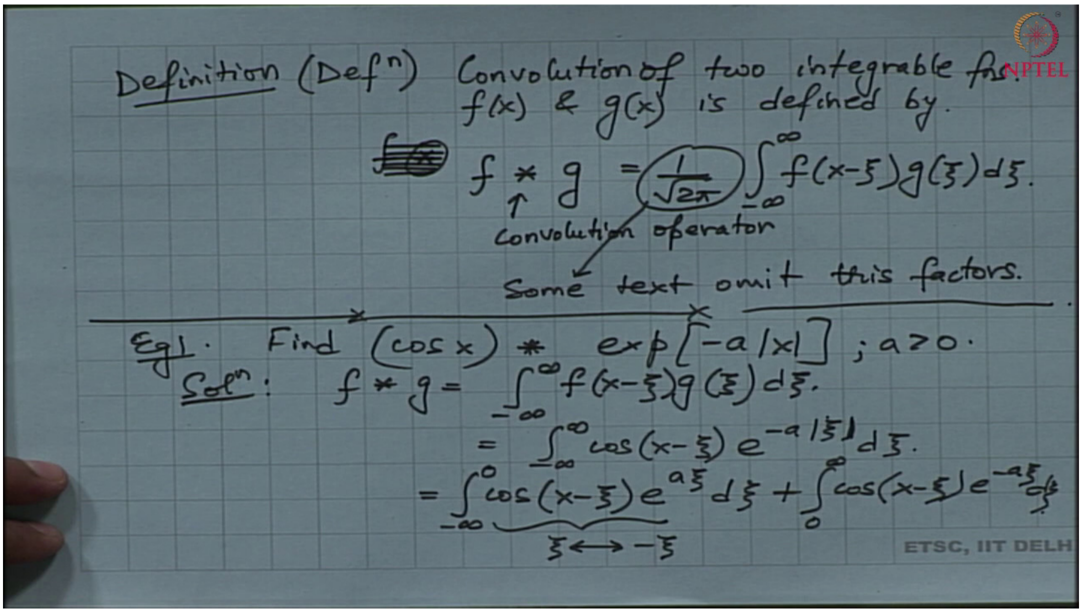
So, let me just show you the result quickly. Proof:

$$F\left(f'\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f'e^{-ikx}dx$$

So, if I choose first part as my II function and second part as my I function, then I use integration by parts. So, then the result tells me that this is equal to

$$= (ik)\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}f(x)dx\right].$$

And I notice that this is nothing, but the Fourier transform of your function f and hence the result. So, that is a very useful result to have.



So, then I have another definition that I want to introduce. So, in short from now I am going to say definition as follows. So, the definition is that of convolution. So, in particular if I have convolution of two integrable functions right given by f of x and g of x is defined by following operator:

$$f * g = \left(\frac{1}{\sqrt{2\pi}}\right) \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi$$

You will see that some text they omit this factor so, some text omit this factor of $\frac{1}{\sqrt{2\pi}}$ So, the students should not be confused if you do not see this factor around, but the convolution is defined as this infinite integral which is the integral over this dummy variable ok.

Example: Find $(\cos x) * \exp[-a|x|]; a > 0$

Solution:

$$f * g = \int_{-\infty}^{\infty} f(x-\xi) g(\xi) d\xi$$

$$= \int_{-\infty}^{\infty} \cos(x-\xi) e^{-a|\xi]} d\xi$$

$$= \int_{-\infty}^{0} \cos(x-\xi) e^{a\xi} d\xi + \int_{0}^{\infty} \cos(x-\xi) e^{-a\xi} d\xi$$

$$= \int_{0}^{\infty} \cos(x+\xi) e^{-a\xi} d\xi + \int_{0}^{\infty} \cos(x-\xi) e^{-ax} d\xi$$

$$= \int_{0}^{\infty} e^{-a\xi} [\cos(x+i) + \cos(e-\xi)] d\xi$$

$$= 2\cos x \int_{0}^{\infty} e^{-a\xi} \cos\xi d\xi$$

$$= (2\cos x)\left(\frac{a}{1+a^2}\right)$$

$$= \int_0^\infty \cos(x+\xi)\, e^{-a\xi}\, d\xi + \int_0^\infty \cos(x-\xi)\, e^{-a\xi}\, d\xi.$$

$$= \int_0^\infty e^{-a\xi} \underbrace{\left[\cos(x+\xi) + \cos(x-\xi)\right]}_{2\cos x \, \cos\xi.} d\xi.$$

$$= 2\cos x \underbrace{\underbrace{\int_0^\infty e^{-a\xi} \cos\xi \, d\xi}_{\text{Int. by parts.}}}_{\left(\frac{a}{1+a^2}\right).}$$

$$= (2\cos x)\left(\frac{a}{1+a^2}\right).$$