Integral Transforms and Their Applications

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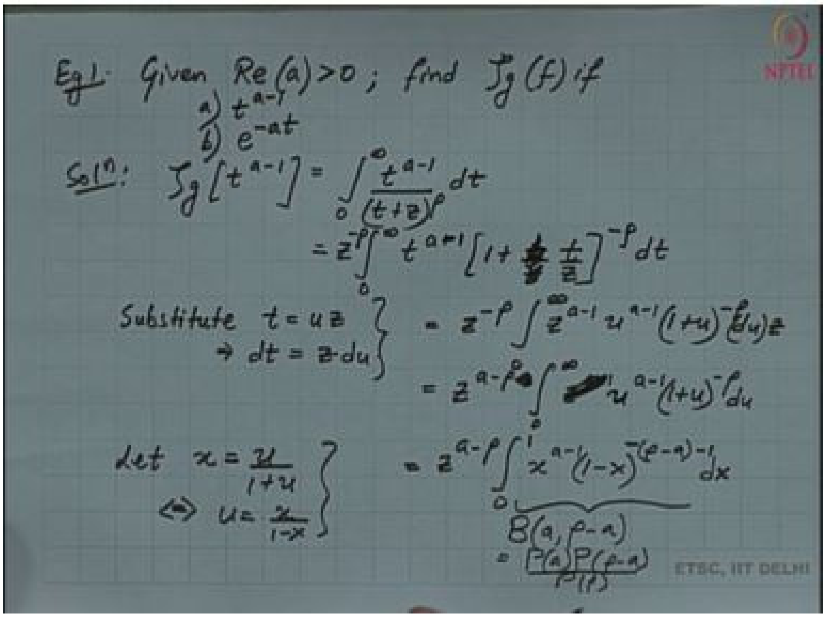
Lecture –13

Applications of Stieltjes Transform, Generalized Stieltjes Transform Part - 03

So, I am given that the real of real value of a number is positive and I need to find. So, I am given a number and the number is this constant is real and positive and I have to find the Stieltjes transform,



Example 1: Given $\operatorname{Re}(a) > 0$; find $\mathcal{S}_g(f)$ if,

a) t^{a-1}
b) e^{-at}

Solution:

$$\mathcal{S}_g\left[t^{a-1}\right] = \int_0^\infty \frac{t^{a-1}}{(t+z)^\rho} dt$$

$$= z^{-\rho} \int_0^\infty t^{a-1} \left[1 + \frac{t}{z}\right]^{-\rho} dt$$

Again I am going to substitute. So, let us substitute,$t = uz$ then we got $dt = zdu$

$$= z^{-\rho} \int_0^\infty z^{a-1} u^{a-1} (1+u)^{-\rho} duz$$

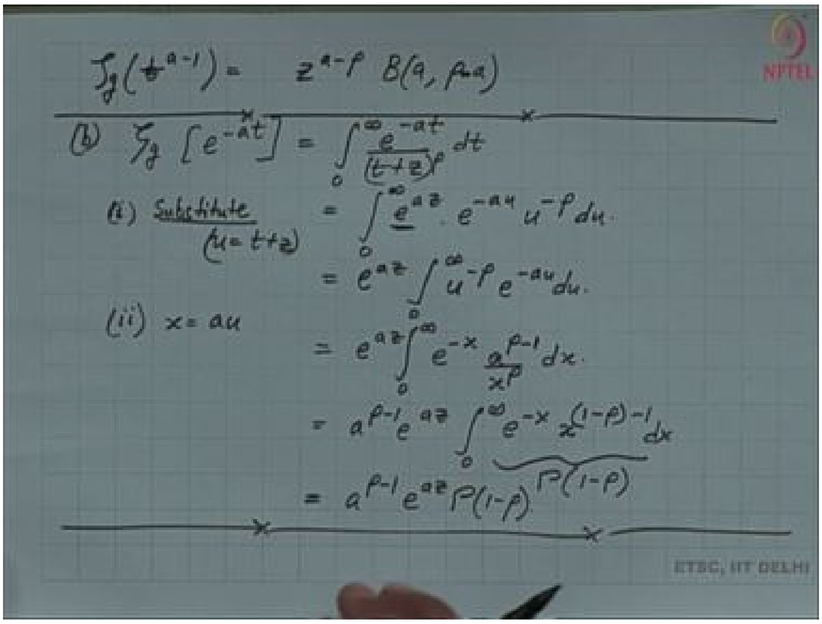
$$= z^{a-\rho} \int_0^\infty u^{a-1} (1+u)^{-\rho} du$$

$$= z^{a-\rho} \int_0^1 x^{a-1} (1-x)^{-(\rho-a)-1} dx$$

$$\text{Let } x = \frac{u}{1+u}$$
$$\Leftrightarrow u = \frac{x}{1-x}$$

then we got,

$$= \frac{\Gamma(a)\Gamma(\rho - a)}{\Gamma(\rho)}$$



$$\mathcal{S}_g f\left(t^{a-1}\right) = z^{a-\rho} B(a, \rho - a)$$

b)

$$\mathcal{S}_g\left[e^{-at}\right] = \int_0^\infty \frac{e^{-at}}{(t+z)^\rho} dt$$

(i).Substitute:(u=t+z);

$$= \int_0^\infty e^{az} \cdot e^{-au} u^{-\rho} du$$

$$= e^{az} \int_0^\infty u^{-\rho} e^{-au} du$$

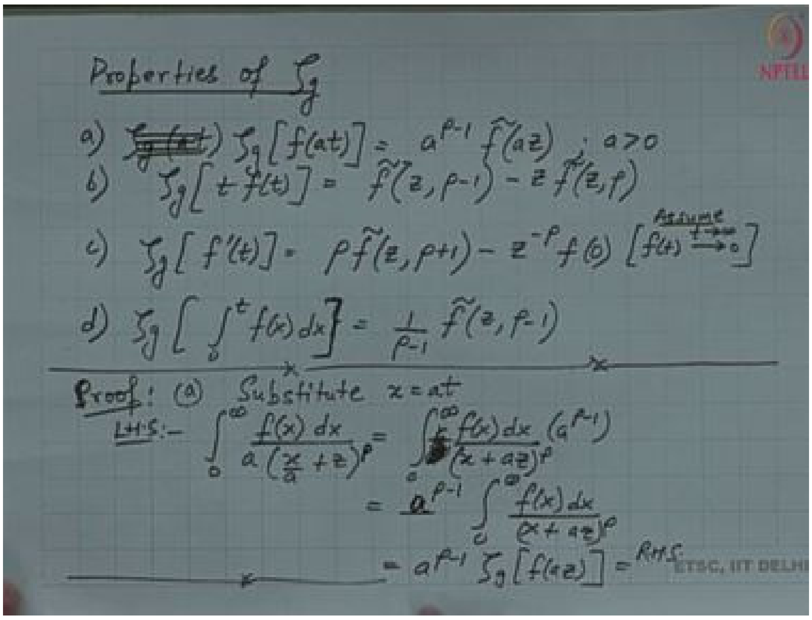
(ii).Substitute:x=au;

$$= e^{az} \int_0^\infty e^{-x} \frac{a^{\rho-1}}{x^p} dx$$

$$= a^{\rho-1} e^{az} \int_0^\infty e^{-x} x^{(1-\rho)-1} dx$$

$$= a^{\rho-1} e^{az} \Gamma(1-p)$$

So, then let us move ahead, I have few more properties of the Stieltjes transform which I want to highlight.



Properties of $\mathcal{S}_g$:

a)

$$\mathcal{S}_g[f(at)] = a^{\rho-1}f(az) \qquad ;a>0$$

b)

$$\mathcal{S}_g[tf(t)] = f(z,\rho-1) - zf(z,\rho)$$

c)

$$\mathcal{S}_g\left[f'(t)\right] = \rho\tilde{f}(z,\rho+1) - z^{-\rho}f(0)$$

d)

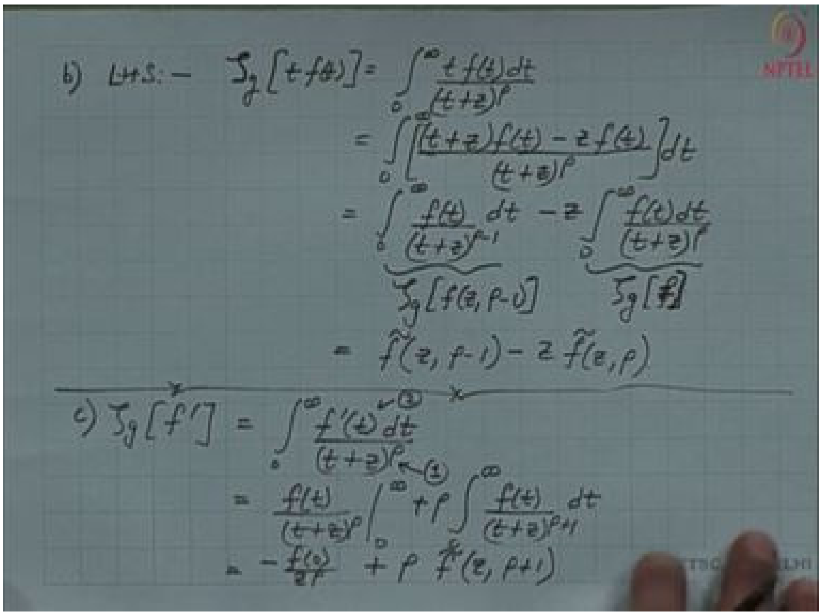
$$\mathcal{S}_g\left[\int_0^t f(x)dx\right] = \frac{1}{\rho-1}\tilde{f}(z,\rho-1)$$

Proof: a) Substitute x=at,

$$L\cdot H\cdot S = \int_0^\infty \frac{f(x)dx}{a\left(\frac{x}{a}+z\right)^\rho} = \int_0^\infty \frac{f(x)dx}{(x+az)^\rho}\left(a^{\rho-1}\right)$$

$$= a^{\rho-1}\int_0^\infty \frac{f(x)dx}{(x+az)^p}$$

$$= a^{\rho-1}\mathcal{S}_g[f(az)] = R\cdot H\cdot S.$$



b)

$$L \cdot H \cdot S = \mathcal{S}_g[tf(t)] = \int_0^\infty \frac{tf(t)dt}{(t+z)^\rho}$$

$$= \int_0^\infty \left[\frac{(t+z)f(t) - zf(t)}{(t+z)^\rho} \right] dt$$

$$= \int_0^\infty \frac{f(t)}{(t+z)^{\rho-1}} dt - z \int_0^\infty \frac{f(t)dt}{(t+z)^\rho}$$

$$= \tilde{f}(z, \rho - 1) - z\tilde{f}(z, \rho)$$

c)

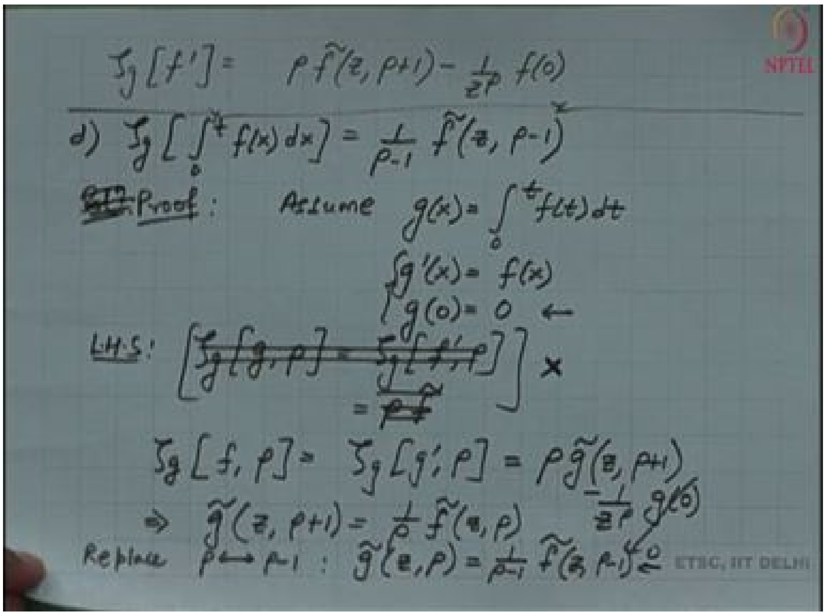
$$\int_g [f'] = \int_0^\infty \frac{f'(t)dt}{(t+z)^\rho}$$

Solving using integration by part,

$$= \left. \frac{f(t)}{(t+z)^\rho} \right|_0^\infty + \rho \int_0^\infty \frac{f(t)}{(t+z)^{\rho+1}} dt$$

$$= -\frac{f(0)}{z^\rho} + \rho\tilde{f}(z, \rho + 1)$$

$$\mathcal{S}_g [f'] = \rho\tilde{f}(z, \rho + 1) - \frac{1}{z^\rho} f(0)$$



d)

$$\mathcal{S}_g\left[\int_0^t f(x)dx\right] = \frac{1}{\rho-1}\tilde{f}(z,\rho-1)$$

Proof:

$$\text{Assume },g(x) = \int_0^t f(t)dt$$

$$\left\{\begin{array}{l} g'(x) = f(x) \\ g(0) = 0 \end{array}\right\}$$

Taking Left Hand Side,

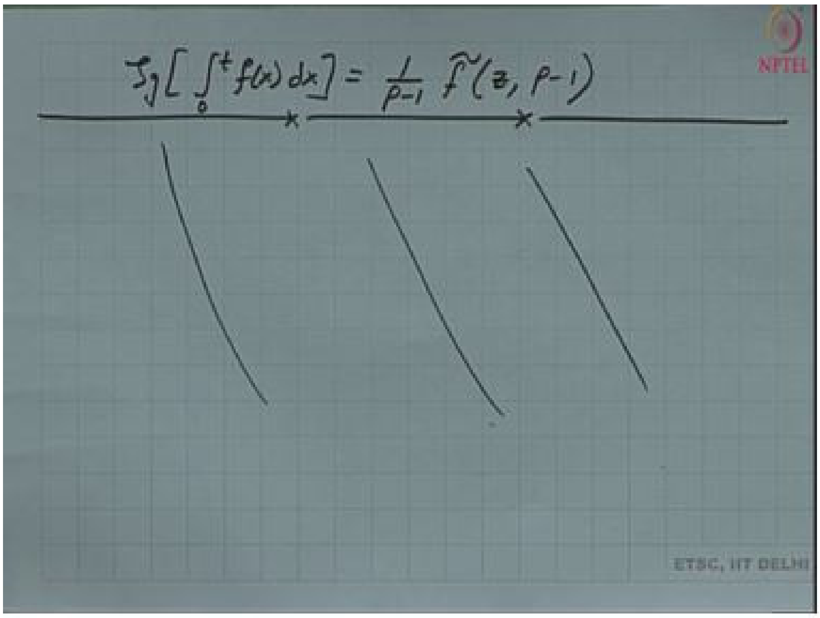
$$\mathcal{S}_g[f,\rho] = \mathcal{S}_g[g',\rho] = \rho\tilde{g}(z,\rho+1) - \frac{1}{z^\rho}g(0)$$

$$\Rightarrow \tilde{g}(z,\rho+1) = \frac{1}{\rho}\tilde{f}(z,\rho)$$

Replace $\rho \to \rho - 1$,

$$\tilde{g}(z,\rho) = \frac{1}{\rho-1}f(z,\rho-1)$$

$$\mathcal{S}_g\left[\int_0^t f(x)dx\right] = \frac{1}{\rho-1}\tilde{f}(z,p-1)$$



So, these are some of well I conclude my discussion on Stieltjes transform in this lecture. In the next lecture, I am going to highlight and start with another new transform known as the Legendre transform. We will see that the Legendre transforms are specially useful to evaluate and solve certain pds involving elliptic operators. For example Laplacians and so on and to evaluate Legendre transform, we will use the famous Legendre polynomial as well as the typical all the properties including the recursion of the Legendre functions recursion of its derivatives as well as of its integral. So, thank you very much.