Integral Transforms and Their Applications

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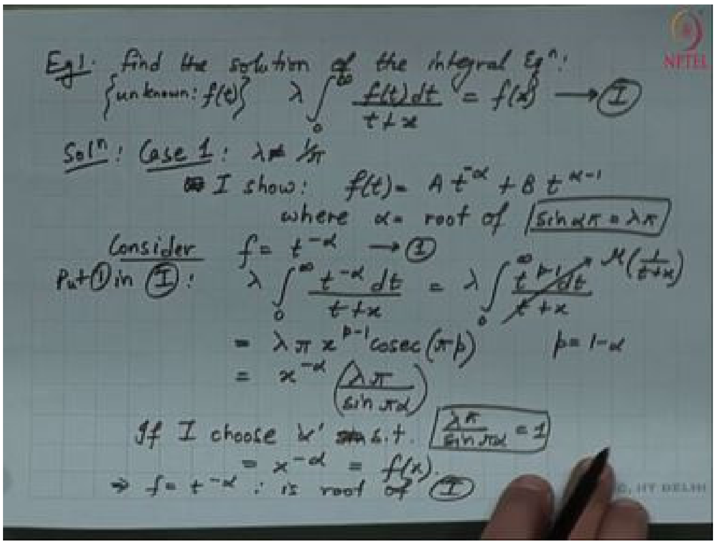
Lecture – 13
Applications of Stieltjes Transform, Generalized Stieltjes Transform Part - 02

Example 1)The example says find the solution find the solution of the integral find the solution of the integral equation given by:

$$\lambda \int_0^\infty \frac{f(t)dt}{t+x} = f(x) \qquad ....(1)$$



So, the solution consists of two cases. We see that if lambda assumes a particular value then we can right away solve using the definition of Stieltjes transform, but if not then we have to process the solution a bit further. So, let me just divide it into two cases.

Case 1).

$$\lambda \neq \frac{1}{\pi}$$

So, I am going to show that,

$$f(t) = At^{-\alpha} + Bt^{\alpha-1}$$

where $\alpha$is the root of $\sin \alpha\pi = \lambda\pi$

Consider,

$$f = t^{-\alpha} \rightarrow (a)$$

Now Substitute equation (a) in equation (1),

$$\lambda \int_0^\infty \frac{t^{-\alpha}dt}{t+x} = \lambda \int_0 \frac{t^{p-1}dt}{t+x}$$

where ,$p = 1 - \alpha$

$$= \lambda \pi x^{p-1} cosec(\pi p)$$

$$= x^{-\alpha} \left(\frac{\lambda \pi}{\sin \pi \alpha} \right)$$

Now we see that if I choose my α such that I have,

$$\left(\frac{\lambda \pi}{\sin \pi \alpha} \right) = 1$$

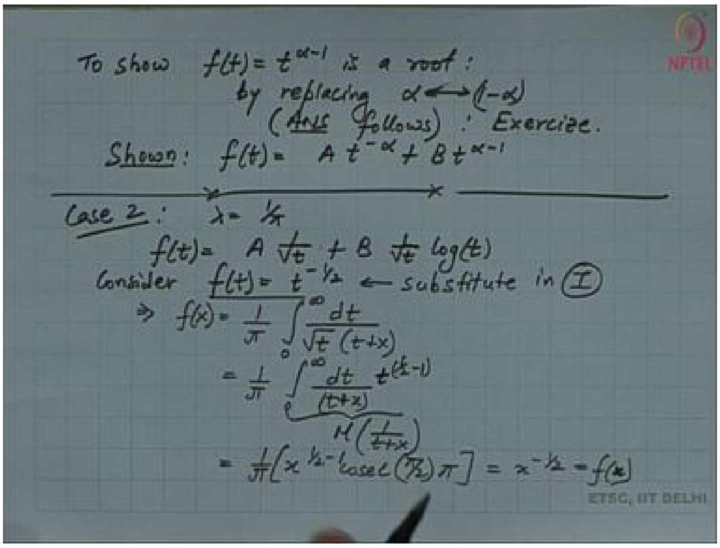
$$= x^{-\alpha} = f(x)$$

$$f = t^{-\alpha} \text{:is the root of(1)}$$

So, what I want to highlight is to show that if to show that the other function given by $t^{\alpha-1}$ is a root I have the following I have that. I can easily show that this function is also the root by replacing my α by $1 - \alpha$ in the previous expression of my steps. So, when I do that I can so, the answer will follow right away. So, I leave this as an exercise to the students to see that, if I replace my α by $1 - \alpha$ then I also get that f(t) given by this $t^{\alpha-1}$ is another root.

So, that completes the first case. So, I have shown that :

$$f(t) = At^{-\alpha} + Bt^{\alpha-1}$$



So, moving on let us look at another case,
Case 2).

$$\lambda = \frac{1}{\pi}$$

$$f(t) = A \frac{1}{\sqrt{t}} + B \frac{1}{\sqrt{t}} \log(t)$$

$$\text{Consider, } f(t) = t^{-1/2}$$

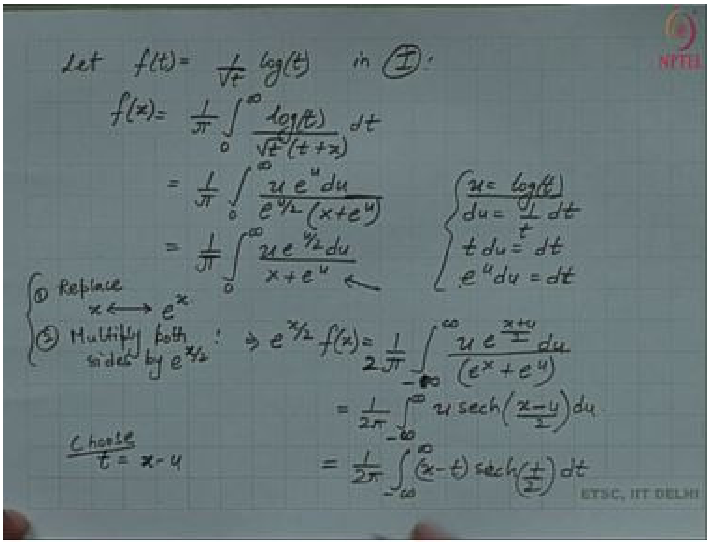
Substitute in Equation(1),

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{dt}{\sqrt{t}(t+x)}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{dt}{(t+x)} t^{(\frac{1}{2}-1)}$$

$$= \frac{1}{\pi}\left[x^{\frac{1}{2}-1} - cosec(\frac{\pi}{2})\pi\right]$$

$$= x^{-1/2} = f(x)$$



$$\text{Let } f(t) = \frac{1}{\sqrt{t}}\log(t) \text{... in (1)}$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\log(t)}{\sqrt{t}(t+x)} dt$$

$$= \frac{1}{\pi}\int_0^\infty \frac{ue^u du}{e^{u/2}\,(x+e^u)}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{ue^{\frac{u}{2}} du}{x+e^u}$$

Where,

$$u = \log(t)$$
$$du = \tfrac{1}{t} dt$$
$$tdu = dt$$
$$e^u du = dt$$

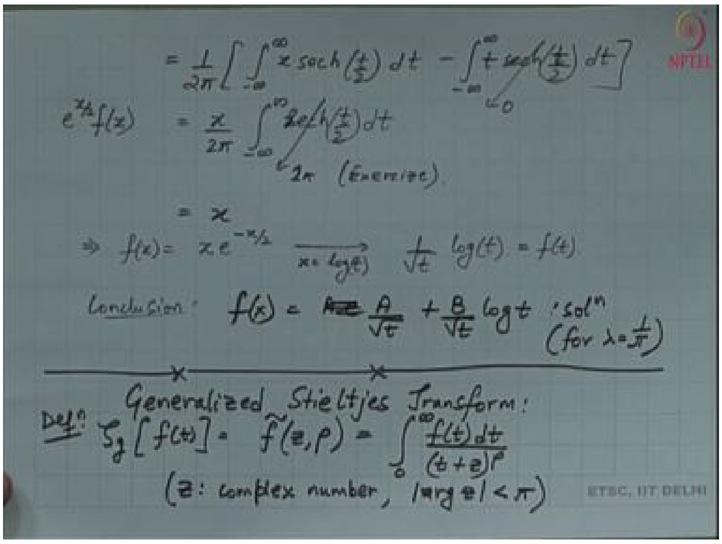
So, these are the steps I will follow. So, first is I am replacing in this expression given here I am replacing x by e^x, and I am also multiplying both sides by $e^{\frac{x}{2}}$.

$$e^{x/2}f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{ue^{\frac{x+u}{2}}du}{(e^x+e^u)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} u\sec h\left(\frac{x-u}{2}\right)du$$

choose $t = x - u$,

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}(x-t)sech\left(\frac{t}{2}\right)dt$$



$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} x\,\text{sech}\left(\frac{t}{2}\right)dt - \int_{-\infty}^{\infty} t\,\text{sech}\left(\frac{t}{2}\right)dt\right]$$

$$e^{x/2}f(x) = \frac{x}{2\pi}\int_{-\infty}^{\infty}\text{sech}\left(\frac{t}{2}\right)dt$$

$$= x$$

$$f(x) = xe^{-x/2} \rightarrow \frac{1}{\sqrt{t}}\log(t) = f(t)$$

$$\text{Conclusion,} f(x) = \frac{A}{\sqrt{t}} + \frac{B}{\sqrt{t}}\log t$$

So, so far I have already discussed Hilbert transform and then the specific case of Hilbert transform and now I am going to generalize my results for Stieltjes transform and look at the generalized transform ok. So, let me just define the generalized Stieltjes transform it is defined as, Definition:

$$\mathcal{S}_g[f(t)] = \tilde{f}(z, \rho) = \int_0^\infty \frac{f(t)dt}{(t+z)^\rho}$$

in this definition I have assumed that z is a complex number I have assumed that z is a complex number and such that my argument of this number is less than π. So, I need all the principle values of the well I need to make sure that the function is well defined that is hence this particular limitation. So, let us look at some examples.