Integral Transforms and Their Applications

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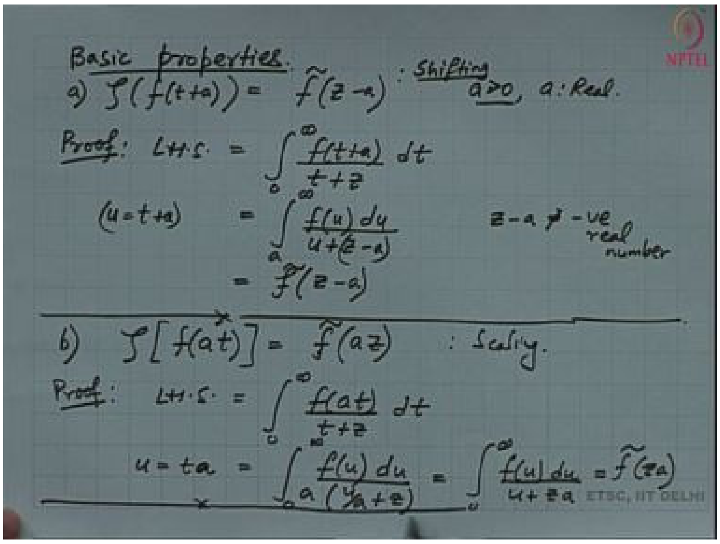
Lecture – 13
Applications of Stieltjes Transform, Generalized Stieltjes Transform Part - 01

So, good afternoon. So, today I am going to continue to discuss our results on Stieltjes transform. Namely I am going to discuss some basic properties of Stieltjes transform and using those basic properties, I am going to talk about and solve some application problems. So, let us continue our discussion.



So, let us look at some properties of Stieltjes transform. So, the first set of properties shows that :

a).

$$\mathcal{S}(f(t+a)) = \tilde{f}(z-a)$$

So, I am going to well the assumption here is that a is positive and a is real as well. So, to see this result; this has a quick proof. So, if I were to let us start with the left hand side.

Proof:

$$L.H.S = \int_0^\infty \frac{f(t+a)}{t+z}dt$$

$$\text{Let}(u = t+a)$$

$$= \int_a^\infty \frac{f(u)du}{u+(z-a)}$$

Now we note that since a is positive, then we must make sure that $z-a$ should not be a negative real number, otherwise the Stieltjes transform is not defined and this was one of the results that I have shown in my last lecture.

$$= \tilde{f}(z-a)$$

b).

$$\mathcal{S}[f(at)] = \tilde{f}(az) \qquad : \text{Scaling}$$

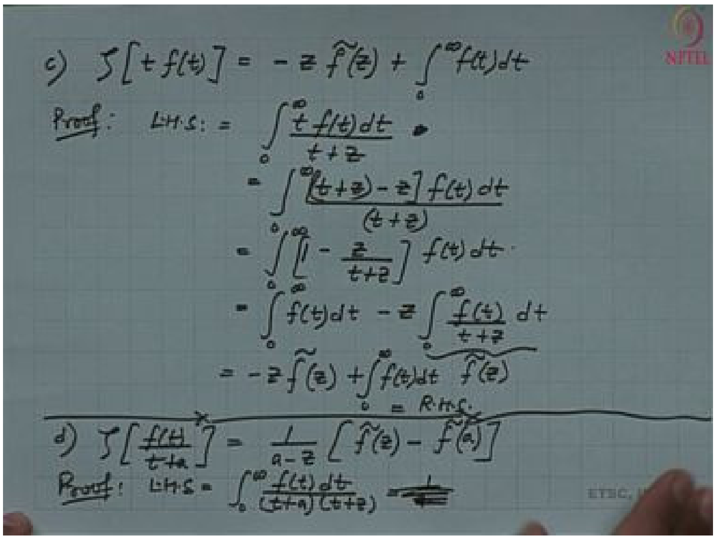
Proof:

$$L \cdot H \cdot S = \int_0^\infty \frac{f(at)}{t+z} dt$$

Let, $\quad u = ta$

$$= \int_0^\infty \frac{f(u)du}{a(u/a + z)}$$

$$= \int_0^\infty \frac{f(u)du}{u + za} = \tilde{f}(za)$$



c).

$$\mathcal{S}[tf(t)] = -z\tilde{f}(z) + \int_0^\infty f(t)dt$$

Proof:

$$L \cdot H \cdot S := \int_0^\infty \frac{tf(t)dt}{t+z}$$

$$= \int_0^\infty \frac{[(t+z)-z]f(t)dt}{(t+z)}$$

$$= \int_0^\infty \left[1 - \frac{z}{t+z}\right] f(t)dt$$

$$= \int_0^\infty f(t)dt - z\int_0^\infty \frac{f(t)}{t+z}dt$$

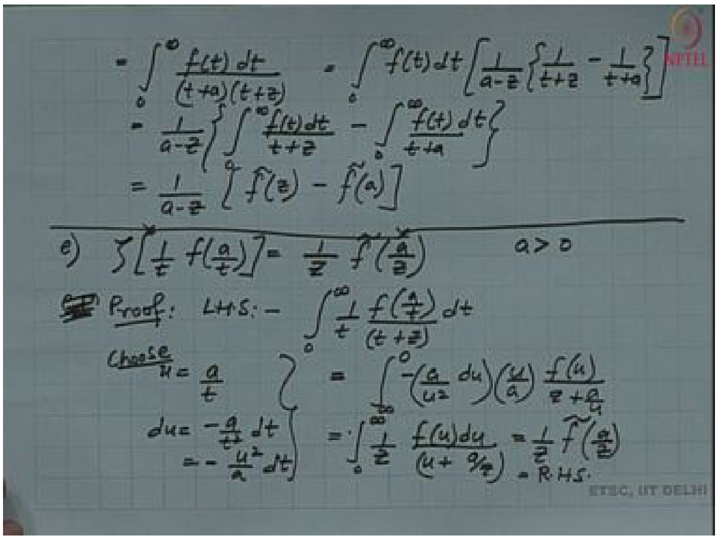
$$= -z\tilde{f}(z) + \int_0^\infty f(t)dt = R \cdot H \cdot S$$

d).

$$\mathcal{S}\left[\frac{f(t)}{t+a}\right] = \frac{1}{a-z}[\tilde{f}(z) - \tilde{f}(a)]$$

Proof:

$$L \cdot H \cdot S = \int_0^\infty \frac{f(t)dt}{(t+a)(t+z)}$$



$$= \int_0^\infty \frac{f(t)dt}{(t+a)(t+z)}$$

$$= \int_0^\infty f(t)dt\left[\frac{1}{a-z}\left\{\frac{1}{t+z} - \frac{1}{t+a}\right\}\right]$$

$$= \frac{1}{a-z}\left\{\int_0^\infty \frac{f(t)dt}{t+z} - \int_0^\infty \frac{f(t)dt}{t+a}\right\}$$

$$= \frac{1}{a-z}\{\tilde{f}(z) - \tilde{f}(a)]$$

e).

$$\mathcal{S}\left[\frac{1}{t}f\left(\frac{a}{t}\right)\right] = \frac{1}{z}f\left(\frac{a}{z}\right) \qquad a > 0$$

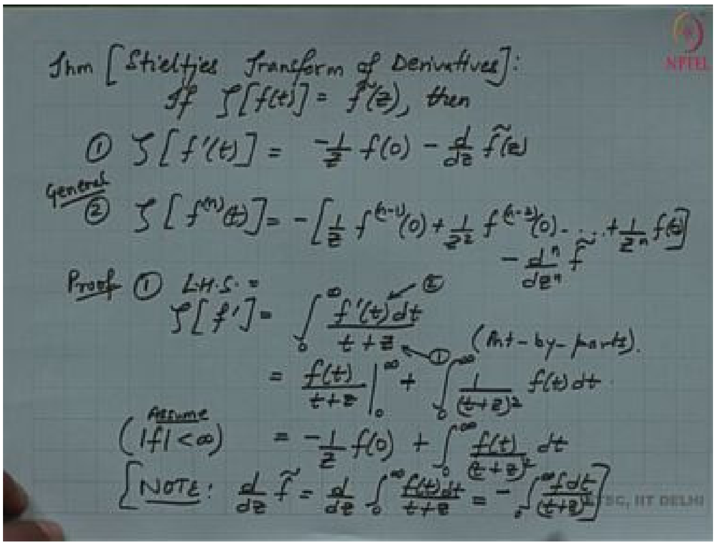
Proof:

$$LHS = \int_0^\infty \frac{1}{t}\frac{f\left(\frac{a}{t}\right)}{(t+z)}dt$$

$$\begin{aligned} u &= \frac{a}{t} \\ \text{Choose, } du &= -\frac{a}{t^2}dt \\ &= -\frac{u^2}{a}dt \end{aligned}$$

$$= \int_{\infty}^{0} -\left(\frac{a}{u^2}du\right)\left(\frac{u}{a}\right)\frac{f(u)}{z+\frac{a}{u}}$$

$$= \int_{0}^{\infty} \frac{f(u)du}{z(u+a/z)} = \frac{1}{z}\tilde{f}\left(\frac{a}{z}\right)$$



So, I stated as a theorem. So, the theorem is related to the Stieltjes transform of derivatives Stieltjes transform of derivatives.

Stieltjes transform of derivatives:

$$\text{If } \mathcal{S}[f(t)] = \tilde{f}(z), \text{ then,I have two results.}$$

1)The first result says that,

$$\mathcal{S}\left[f'(t)\right] = -\frac{1}{z}f(0) - \frac{d}{dz}\tilde{f}(z)$$

And similarly the second in fact, we can construct many such statement for each they for each of the derivative of the function let me state the general result. So, the general result in this theorem says that that ,

2)

$$\mathcal{S}\left[f^n(t)\right] = -\left[\frac{1}{z}f^{(n-1)}(0) + \frac{1}{z^2}f^{(n-3)}(0) + \frac{1}{z^n}f(0)\right] - \frac{d^n}{dz^n}\tilde{f}$$

Proof: 1).

$$L \cdot H \cdot S := \mathcal{S}\left[f'\right] = \int_0^{\infty} \frac{f'(t)dt}{t+z}$$

Now Integration by part,

$$= \left.\frac{f(t)}{t+z}\right|_0^{\infty} + \int_0^{\infty} \frac{1}{(t+z)^2}f(t)dt$$

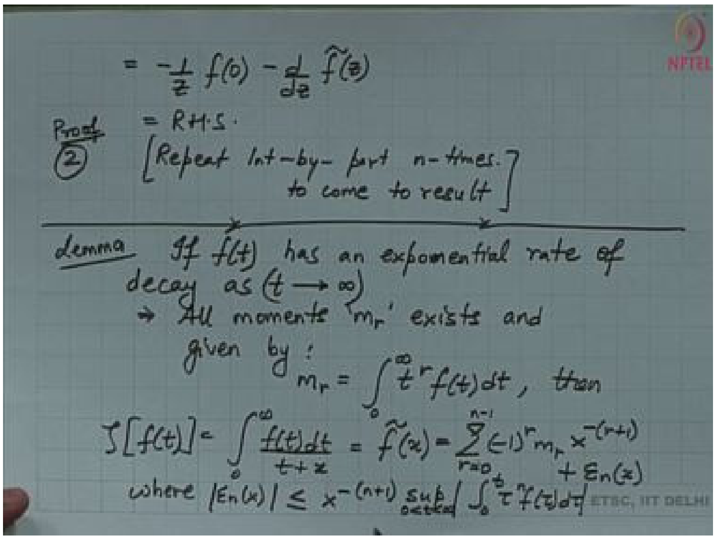
$$\text{Assume,}(|f| < \infty)$$

$$= -\frac{1}{z}f(0) + \int_0^\infty \frac{f(t)}{(t+z)^2}dt$$

$$\text{Note: } \frac{d}{dz}\tilde{f} = \frac{d}{dz}\int_0^\infty \frac{f(t)dt}{t+z} = -\int_0^\infty \frac{fdt}{(t+z)^2}$$

$$= -\frac{1}{z}f(0) - \frac{d}{dz}\tilde{f}(\bar{z}) = R \cdot H \cdot S$$

Now I am going to leave the proof of the second case because the second case is nothing but the first case repeated n times. So, repeat integration by parts n times to come to the result ok. So, this is a hint that I have given to the students and using this hint the students can very easily derive the second rich part of the theorem which is the general case of the nth derivative of Stieltjes transformed.



So, then I have more results. I have another result in the form; let me just stated in the form of a lemma which says that, If f(t) has an expomential rate of decay as,$t \to \infty$

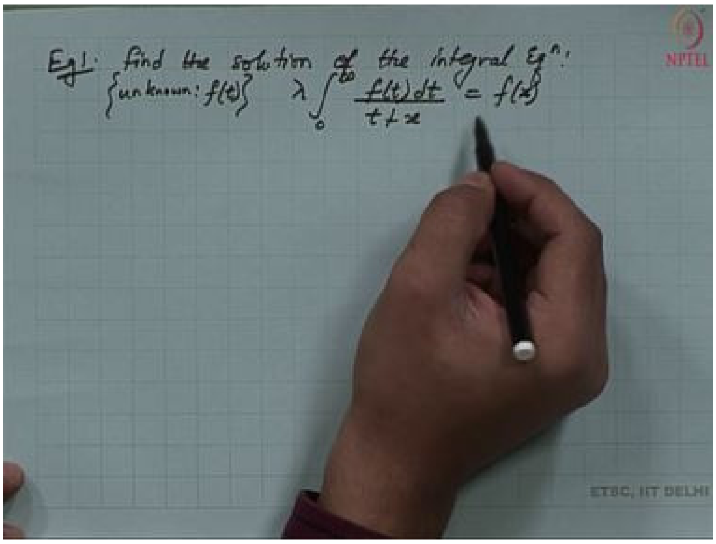
All moments m_r exists and given by:

$$m_r = \int_0^\infty t^r f(t)dt$$

$$\mathcal{S}[f(t)] = \int_0^\infty \frac{f(t)dt}{t+x} = \tilde{f}(x) = \sum_{r=0}^{n-1}(-1)^r \, \mathrm{m}_r \, x^{-(r+1)} + \mathcal{E}_n(x)$$

$$\text{where, } |\mathcal{E}_n(x)| \leqslant x^{-(n+1)} \quad \text{supermum}_{0<t<\infty}| \int_0^\infty \tau^n f(\tau)d\tau$$

So, the supremum is taken over all values of this upper limit of the integral. So, so that is that is the result. So, then I am going to use this result in the following example.



Example 1)The example says find the solution find the solution of the integral find the solution of the integral equation given by:

$$\lambda \int_0^\infty \frac{f(t)dt}{t+x} = f(x)$$

So, we need to solve this.