Integral Transforms and Their Applications
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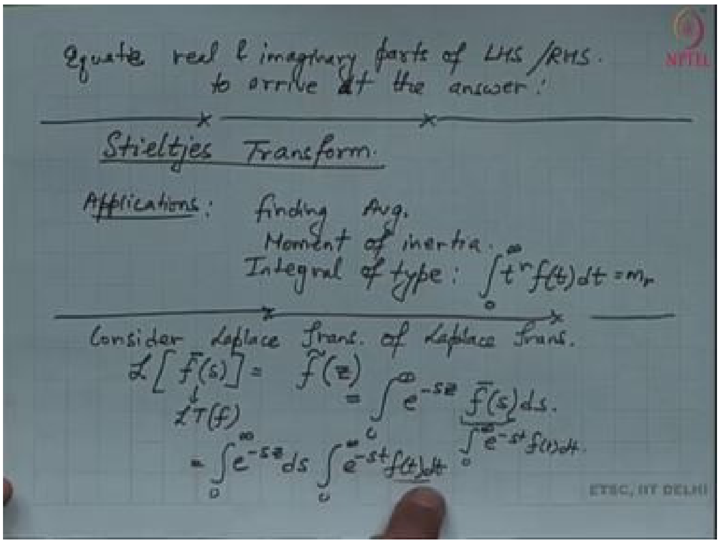
Lecture – 12
Applications of Hilbert Transforms, Introduction to Stieltjes Transforms Part - 03

So, what I said is equate real and imaginary parts of the left hand side and the right hand side of the expression to come at to arrive at the answer. So, that I leave it to the students that is quite straightforward to see that the real part corresponds to the first expression and the imaginary part corresponds to the second quantity that we have to prove.So, moving on I am going to further continue my discussion on 1-sided Hilbert transform especially known as the Stieltjes transform.So, as I have briefly introduced Stieltjes transform is a specific case of one sided Hilbert transform. Now, Stieltjes transform are specially useful in evaluating some problems involving elastic you know elastic domains elasticity problems and specially involving certain moments, moments of the domain.

So, when I say moments. So, it is specially, so, let me just highlight some of the applications. So, the applications involve you know finding certain averages; so, finding certain averages or say in problems related to mechanics it is quite useful in finding moment of inertia or integrals of the form given here;

$$\text{Integral of type: } \int_0^\infty t^r f(t)dt = m_r$$



So, I see that these are my specific integral I call as my moments integral; so, the rth moment.So, to begin with let us consider the Laplace transform of a Laplace transform.

So, what I am saying is consider the Laplace transform of the Laplace transform.

$$\mathcal{L}[\overline{f}(s)] = \tilde{f}(z)$$

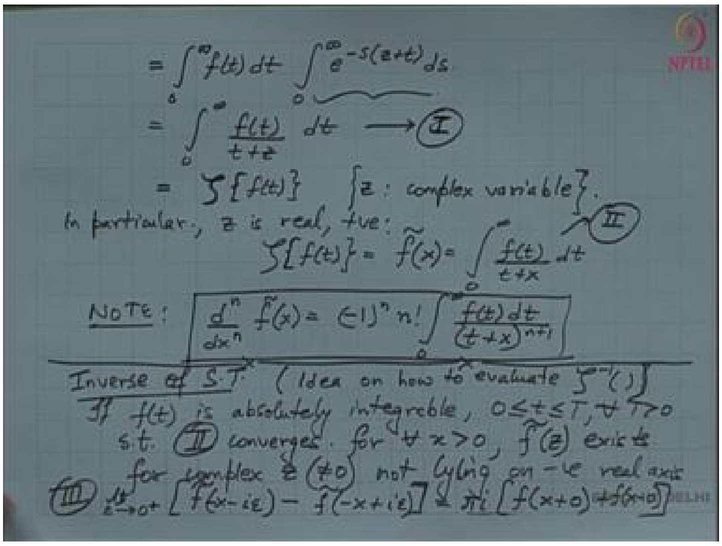
$$= \int_0^\infty e^{-sz}\overline{f}(s)ds$$

$$= \int_0^\infty e^{-sz} ds \int_0^\infty e^{-st} f(t) dt$$

$$= \int_0^\infty f(t) dt \int_0^\infty e^{-s(z+t)} ds$$

$$= \int_0^\infty \frac{f(t)}{t+z} dt \qquad ....(1)$$

$$= \mathcal{S}\{f(t)\}$$



So, in evaluating this transform my z is a complex variable in general. So, then if in particular if my z is real and positive.

$$\mathcal{S}\left[f(t)\right] = f(x) = \int_0^\infty \frac{f(t)}{t+x} dt \qquad ..(2)$$

Note:

$$\frac{d^n}{dx^n}\tilde{f}(x) = (-1)^n n! \int_0^\infty \frac{f(t)dt}{(t+x)^{n+1}}$$

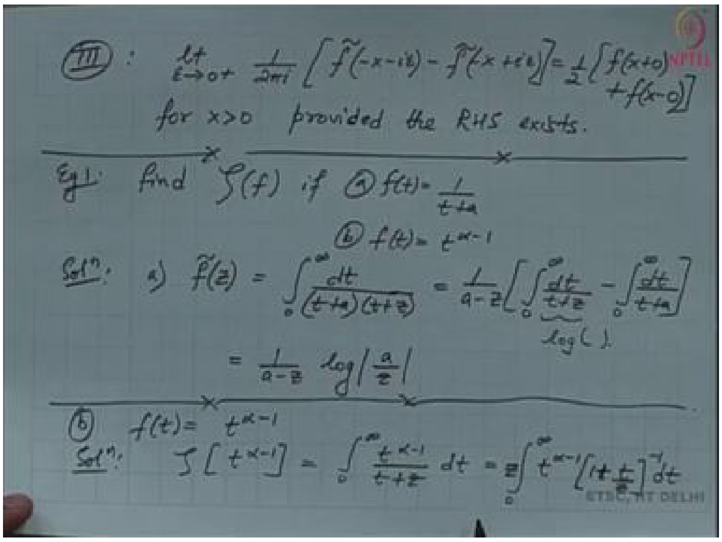
However, I am going to state a result related to the inverse. So, inverse of the Stieltjes transform.

So, it says that if I have a function f which is absolutely integrable, f is absolutely integrable in the domain $0 \leq t \leq T$. So, I fix my domain T for all capital T positive such that such that my expression (2) converges. So, the expression (2) is given by this particular expression that I have highlighted converges for all values x positive, so, all variables positive.

Then my Stieltjes transform of $f(z)$ exists for complex z which is non-trivial scenario not lying on the real axis ok, so, not lying on the negative real axis. So, as we have seen that the Stieltjes transform is defined on the positive half of your complex domain, so, we are taking a z which does not lie on the positive negative real axis where we will have singularities.

So, the result says that

$$\lim_{\varepsilon \to 0^+} [f(-x - i\varepsilon) - f(-x + i\varepsilon)] = \pi i[f(x+0) + f(x-0)]..(3)$$



So, my statement (3) is given by

$$\lim_{\varepsilon\to 0^+}\frac{1}{2\pi i}[\tilde{f}(-x-i\varepsilon)-f(-x+i\varepsilon)]=\frac{1}{2}[f(x+0)+f(x-0)]$$

So, x is the point where we are evaluating the function provided that the right hand side exists, RHS exists. So, we see that this statement gives a relation between the Stieltjes transform and its inverse.So, then let us look at some examples.

Example 1:Find $\mathcal{S}(f)$ if :

$$a)f(t)=\frac{1}{t+a}$$
$$b)f(t)=t^{\alpha-1}$$

Solution:

a).

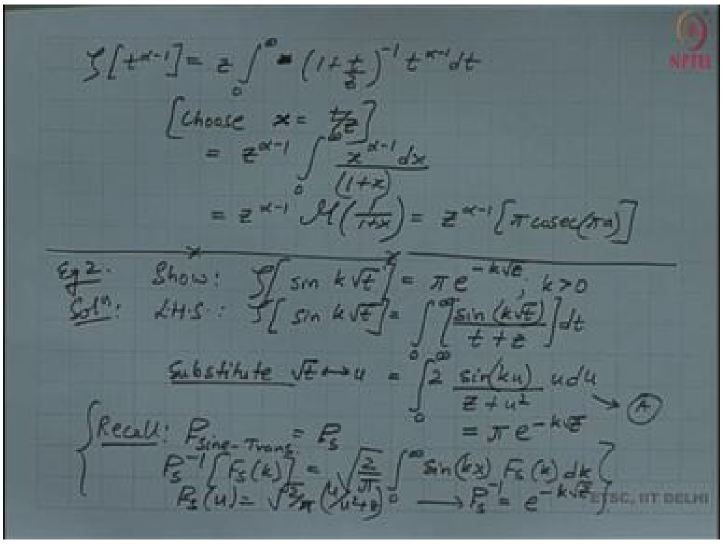
$$\tilde{f}(z)=\int_0^\infty\frac{dt}{(t+a)(t+z)}$$
$$=\frac{1}{a-z}\left[\int_0^\infty\frac{dt}{t+z}-\int_0^\infty\frac{dt}{t+a}\right]$$
$$=\frac{1}{a-z}\log\left|\frac{a}{z}\right|$$

b)

$$f(t)=t^{\alpha-1}$$

Solution:

$$\mathcal{S}\left[t^{\alpha-1}\right]=\int_0^\infty\frac{t^{\alpha-1}}{t+z}dt$$
$$\mathcal{S}\left[t^{\alpha-1}\right]=z\int_0^\infty t^{\alpha-1}\left[1+\frac{t}{z}\right]^{-1}dt$$



$$\mathcal{S}\left[t^{\alpha-1}\right] = z\int_0^\infty \left(1+\frac{t}{z}\right)^{-1} t^{\alpha-1}dt$$

choose $x = \frac{t}{z}$,

$$= z^{\alpha-1}\int_0^\infty \frac{x^{\alpha-1}dx}{(1+x)}$$

$$= z^{\alpha-1}\mathcal{M}\left(\frac{1}{1+x}\right)$$

$$= z^{\alpha-1}[\pi cosec(\pi a)]$$

Example 2: Show

$$\mathcal{S}[\sin k\sqrt{t}] = \pi e^{-k\sqrt{z}} \qquad ; k > 0$$

Solution:

$$L\cdot H\cdot S: \mathcal{S}\{\sin k\sqrt{t}\} = \int_0^\infty \left[\frac{\sin(k\sqrt{t})}{t+z}\right]dt$$

$$\text{Substitute } \sqrt{t} \leftrightarrow u = \int_0^\infty \frac{\sin(ku)}{z+u^2}udu$$

$$= \pi e^{-k\sqrt{z}}$$

So, finally, I would like to highlight the fact that we have seen some basic examples of Hilbert transform and we have also seen some particular case of Hilbert transform namely the 1-sided Hilbert transform or the Stieltjes transform. Now, we have seen some examples of Stieltjes transform and in my next lecture I am going to continue my discussion on Stieltjes transform. Namely, I am going to look at some properties and then later on I am also going to briefly discuss another transform, a transform in the discrete signals known as the Z-transform. Thank you very much.