Integral Transforms and Their Applications
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Lecture – 12
Applications of Hilbert Transforms, Introduction to Stieltjes Transforms Part - 01

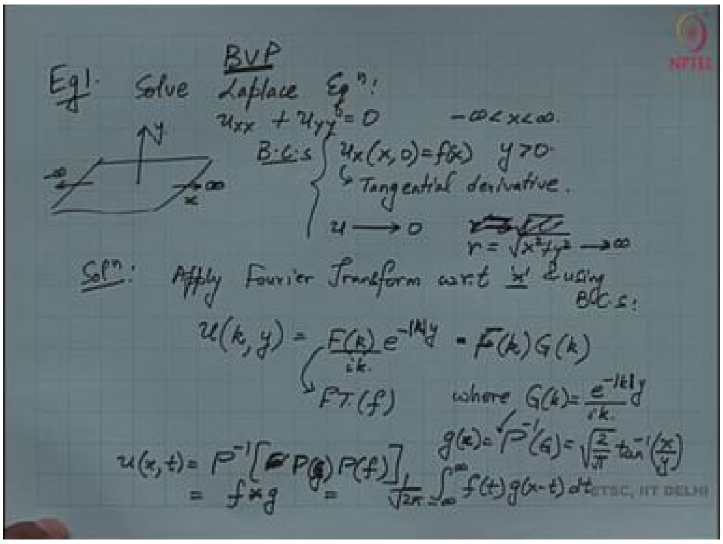
Good morning every one. So, in the last 11 lectures we have introduced and we have seen 5 different types of integral transforms; Fourier transforms, Laplace transforms, Hankel transforms, Mellin transform and very recently the Hilbert transforms. I have also introduced some properties of Hilbert transforms and in today's lecture I am going to discuss about some applications of the Hilbert transforms.

Now, I also want to highlight that we have solved some basic, some standard equations which arise in partial differential equations like the wave equations or the diffusion equation. We have seen equations different types of wave equations for example, the acoustic waves or the waves arising due to the vibration of a medium and so on.

And further we have seen that almost all these waves they have the property that they change their shape as time passes or as the wave progresses in space. We also call these waves as dispersive waves. So, today I am going to discuss the applications of Hilbert transform with regards to a very special type of waves called the solitons. Now these waves are studied by the celebrated Benjamin-Ono equations which I am going to briefly highlight and followed by another soliton equations known as the Korteweg-de Vries equation popularly known as the KdV equations.

Now, so what are these solitons? So, legend has it that there was the physicist by the name of Sir Scott Russell and he was travelling in a canal over the city of Edinburg and what he observed was that there are the certain waves while he was travelling on a boat that certain waves were arising out of the bottom of his boat and travelling far and wide across the canal without changing shape. And what was seen is that so that intrigue the physicist and he went on to study this waves and these waves are famously called as the solitons or the waves which are non dispersive which do not change their shape.

So, I am going to start the applications of Hilbert transforms and later on describe some of the soliton solutions by these equations that I have just mentioned ok.



So, moving on, so I have let us start right away with the example. So, I have an example which is solved. So, this is by the way boundary value problem. So, solve the Laplace equation on a semi infinite domain. So, the equation is of course, is given by we are solving a 2D Laplace equation and the domain is defined as $\infty < x < \infty$ and $y > 0$. So, we are talking about the domain which is like an infinite plate.

$$u_{xx} + u_{yy} = 0 \quad -\infty < x < \infty$$

So, what I see is that this equation and then of course, my boundary conditions are given by:

$$u_x(x,0) = f(x) \quad y > 0$$

So, as we see that this is an equation to be solved on an semi infinite plate well infinite in one direction. So, u_x denotes the tangential derivative of the solution ok. So, then the second equation that I have is the second boundary condition that I need, this is the second order PD. The second boundary condition that I am given is that u decays to 0 as the domain.

So, my domain I define it by this variable r. So, my domain r given by:

$$r = \sqrt{x^2 + y^2} \to \infty$$

so that is the complete description of the problem. So, again if we were to solve this, the standard way to solve this problem is we can apply Fourier transform with respect to x and then we are going to get an ODE.

Solution:we will apply Fourier transform with respect to the variable x and we see that after using both of my boundary conditions I arrive at the following. So, I am going to right away write the solution in the transformed plane.

$$u(k,y) = \frac{F(k)}{ik} e^{-i|k|y} = F(k)G(k)$$

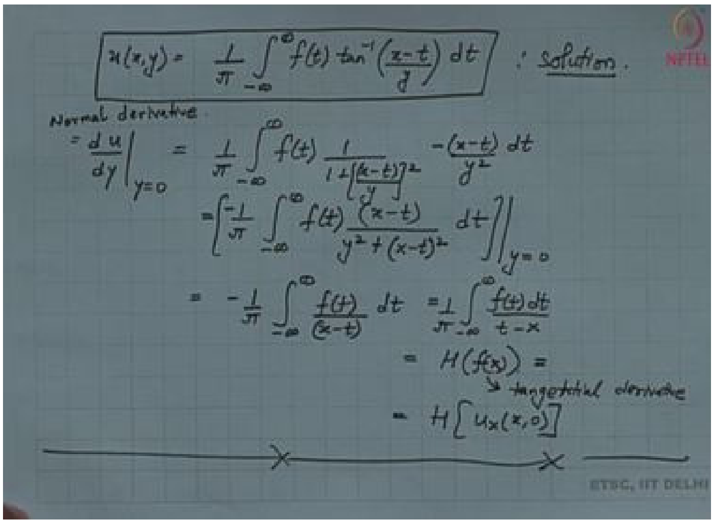
$$\text{where, } G(k) = \frac{e^{-|k|y}}{ik}$$

$$g(x) = \Gamma^{-1}(G) = \sqrt{\frac{2}{\pi}} \tan^{-1}\left(\frac{x}{y}\right)$$

$$u(x,t) = \Gamma^{-1}(\Gamma(g)\Gamma(f))$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)g(x-t)dt$$

$$= f * g$$



$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\tan^{-1}\left(\frac{x-t}{y}\right)dt$$

Normal Derivative,

$$\left.\frac{du}{dy}\right|_{y=0} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{1}{1+\left[\frac{(x-t)}{y}\right]^2} - \frac{(x-t)}{y^2}dt$$

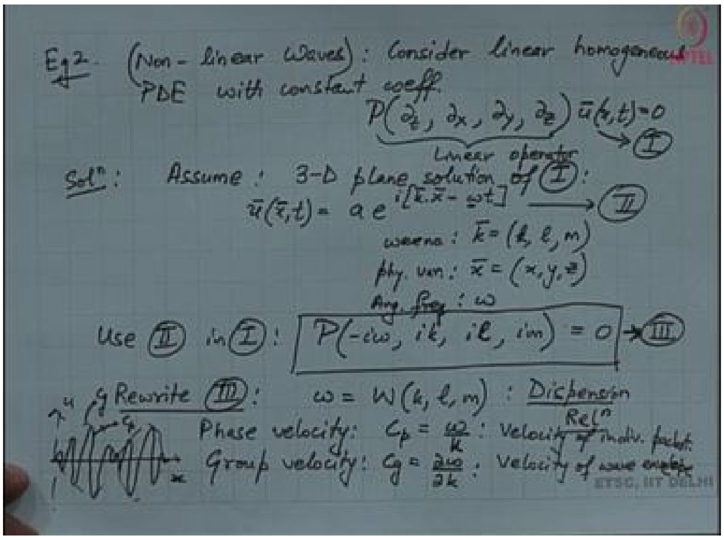
$$= \left[\frac{-1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{(x-t)}{y^2+(x-t)^2}dt\right]\Bigg|_{y=0}$$

$$= -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(x-t)}dt = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)dt}{t-x}$$

$$= H(f(x)) = \text{tangential Derivative}$$

$$= H\left[u_x(x,0)\right]$$

So, moving on, let us now look at some more important applications of Hilbert transform as I had already described in the introduction to this lecture.



Example 2:So, I am going to right away start by describing the solution to non-linear waves ok. So, non-linear waves.

So, to consider this particular equation let us consider the most general equation describing the non-linear waves. So, to see that let us start with considering a linear homogenous PDE; so, linear homogenous PDE with constant coefficient, linier homogeneous PDE with constant coefficients; so, that is the first step of our study of non-linear waves.

So, I can write down the PDE in the most general form. This is a function of the derivative with respect to t. So, let us say that t is my variable time and also a function of with respect to the derivatives of x, with respect to derivatives of x with respect to derivatives of y and with respect to derivatives of z of the solution

$$\mathcal{P}\left(\partial_t, \partial_x, \partial_y, \partial_z\right) \bar{u}(x,t) = 0 \qquad ...(1)$$

So, I can say that this is my linear operator describing the solution. So, operating on this variable u and the result is describing the solution to certain non-linear waves. Now, so the moment I say I am seeking the solution which are showing the wave like features. So, let us assume to begin with that my solution is of the plane wave.

Solution: Assume 3D plane solution of (1).

$$\bar{u}(\bar{x},t) = ae^{i[\bar{k}\cdot\bar{x}-\omega t]} \qquad ...(2)$$

$$\text{wave number} \quad : \bar{k} = (k, \ell, m)$$

$$\text{Physical Variable} \quad : \bar{x} = (x, y, z)$$

$$\text{Angular Frequency:} \quad = \omega$$

Use equation (2) in (1).

$$\mathcal{P}(-i\omega, ik, i\ell, im) = 0 \qquad ..(3)$$

Now Rewrite equation (3):

$$\omega = W(k, l, m)$$

this special form of III is also known as the dispersion relation of the equation, so dispersion relation. Now, I have already briefly mentioned what is this dispersion relation. So, dispersion relation is the relation governing the variables in the transformed planes corresponding to the variables in the physical plane.

So, my variable in the physical plane are x and t, my variable in the transformed plane are k, l, m and ω So, we can consider that the dispersion relation is the transformed version of the original PDE we are trying to solve. Now, most of the most of these cases in most of the cases where we are solving the wave equation the dispersion relation, the relation that is to be satisfied in the transform plane may not be identically equal to the value that we get when we solve the variable in the physical plane.

And hence there is a discrepancy of the solution in the physical and the transformed plane that leads to the so called dispersion in the wave or changes in the shape of the wave ok.

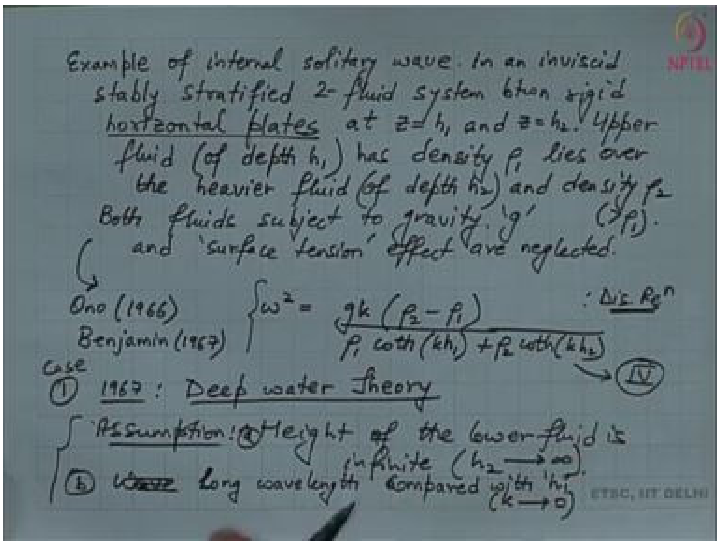
So, again let me just briefly highlight some of the concepts I had already introduced when I was discussing some examples in Fourier transform, namely the concept of phase velocity; the concept of phase velocity and the concept of group velocity.

$$\text{Phase Velocity} : c_p = \frac{w}{k} \qquad \text{,velocity of individual packet}$$

$$\text{Group Velocity} : c_g = \frac{dw}{dk} \qquad \text{,velocity of wave envelope}$$

(Refer the above slide)

So, that is a very naive way to describe the group velocity or the phase velocity via a simple example.so moving on.



So, I am going to talk about a soliton, a soliton waves. So, to do that let us look at a case of. So, without going into an extreme detailed background let me just highlight the example first. So, I am going to right away come to the dispersion relation of this example.

So, the example is that of an internal solitary wave, internal solitary wave in an inviscid, in an inviscid stably stratified stably stratified. So, all these terms are quite applicable for problems in relation to the study of water waves.So, stratified 2-fluid system between rigid horizontal plates.So, what we are studying is a 2-fluids scenario in between two horizontal plates.

Let us say that the plates are situated at height $z = h_1$ and $z = h_2$ and further to distinguish between the two fluids let us say that the upper fluid, the upper fluid of well of depth h_1 which we have already mentioned, of depth h_1 has density ρ_1. So, I am going to distinguish between the upper and lower fluid.

So, and it lies over the heavier fluid, it lies over the heavier fluid of let us say of depth h_2 and density ρ_2. So, these are my variables for the two fluids setup that we have and the fluids are located between the two horizontal or two flat plates and the height of the fluids are respectively h_1 and h_2. So, of course, my density ρ_2 is greater than ρ_1 since the heavier fluid is at the bottom and I am going to assume that both fluids are subject to gravity as well. So, both fluids subject to gravity; so, I have the effect of g and I am going to neglect the surface tension.

So, surface tension effect between the two fluids and otherwise surface tension effect are neglected.So, for that I am going to right away state the result via the dispersion relation. So, this sort of a problem has been already published and worked upon by several physicists several scientists starting from the celebrated scientists Ono in 1966 in his theory and separately by another scientist Benjamin in 1967.

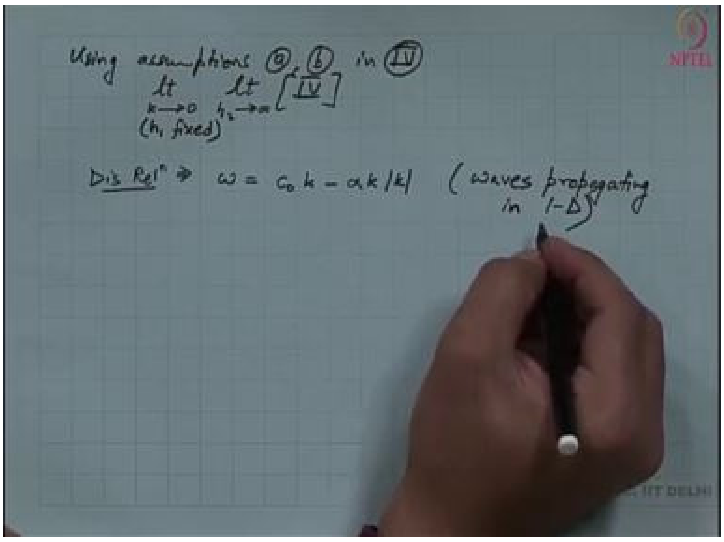
So, celebratory called as the Benjamin-Ono equation. So, the dispersion relation for such an equation for the stratified fluid is given by this following expression.

$$\omega^2 = \frac{gk\left(\rho_2 - \rho_1\right)}{\rho_1 \coth\left(kh_1\right) + \rho_2 \coth\left(kh_2\right)} \qquad ...(4)$$

So, then there were several specific cases that were described for related to this dispersion relation; so, this is my dispersion relation. So, several specific cases were described by these two people, one of the case was described by Benjamin in 1967 also called as the deep water theory. So, as the name suggests we, in this particular example we assume that my depth of the heavier fluid, the fluid which is sitting at the bottom the depth h_2 goes to infinity. So which means the assumption in this case,

so case 1; the assumption is that the height of the lower fluid the height of the lower fluid is infinite the height of the lower fluid is infinite. So, h_2 goes to infinity ok, further there is a second assumption. So, let us call it as a and b that the waves are long compared to the depth h_1 of the fluid ok. So, what it means further? The second assumption further exaggerates the fact or exemplifies the fact that the height of the first fluid of height h_1 is small compared to the height of the second fluid. So, what the second assumption is? Long wave length, long so long wavelengths compared with h_1 ok. So, I assume this is the situation k going to 0 ok.

So, long wavelength corresponds to small wave number as we have seen in some of the reciprocal theorems, like Tauberian theorems ok. So, then under these two assumptions let me call this expression that I have written on top as my expression (4).



So, if I use my assumption using assumptions a and b in (4) which means that we have to take two limits we have to take. So, this is my expression (4), I have to take one limit, the first limit is h_2 going to infinity. So, the depth of the first the first fluid or the heavier fluid sorry the second fluid is going to infinity and then the second situation is limit k going to 0 and I fix for a fixed h_1.

So, compared with the height of the first fluid the wavelengths of the water waves are long.So, the height of the first fluid is negligibly small compared to the wavelength generated from the wave. So, when I do that I take the necessary limit and expand the cot hyperbolic in the expression (4), I get my following results. So, I am going to directly write away the result. So, I get the following dispersion relation;

$$\omega = c_0 k - \alpha k|k|$$

So, here I have assumed. So, this is the most basic case waves propagating in 1D; waves propagating in 1D.