Integral Transforms and Their Applications
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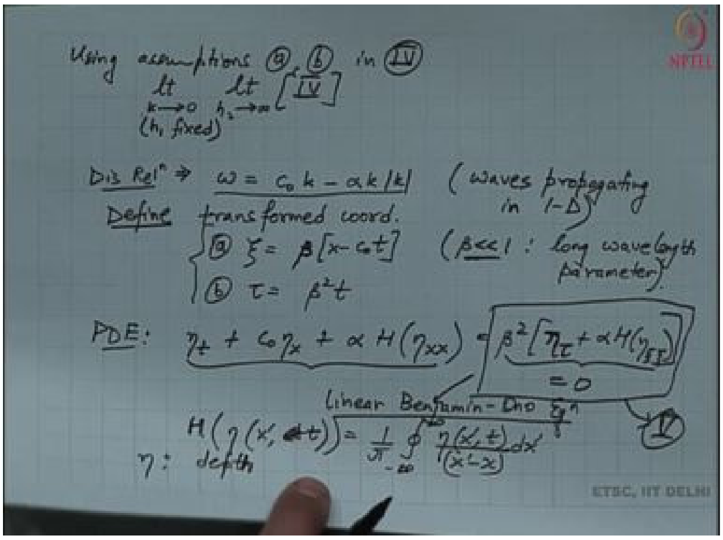
Lecture – 12
Applications of Hilbert Transforms, Introduction to Stieltjes Transforms Part - 02

So then from this dispersion relation I have to I am now going to come back to the original PDE that we are solving. So, to do that let us define a transformed coordinate; so, transformed coordinate as follows:

$$\xi = \beta[x - c_o t]$$



where I assume my beta to be much much small very small or this is nothing, but the same as my long wavelength assumption. So, it is a this particular assumption is arising due to our long wavelength assumption to begin with ok. So, long wavelength let me call this as a long wavelength parameter ok.

$$\tau = \beta^2 t$$

So, when I use these transformed variable and convert this dispersion relation back into my PDE in the physical plane, I get the following PDE:

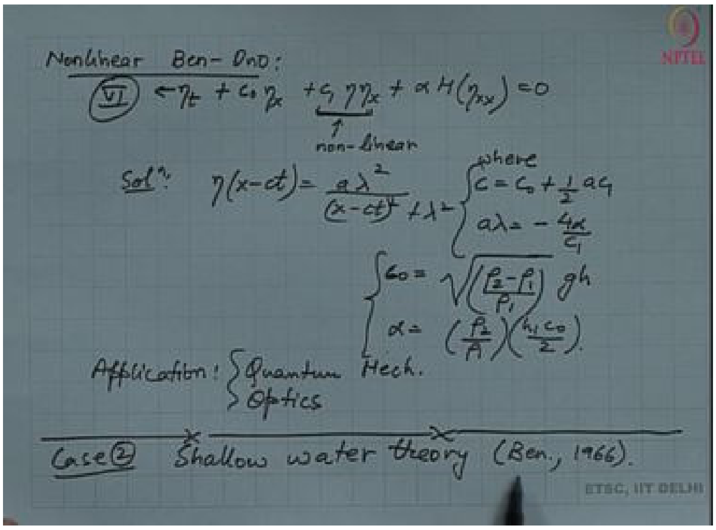
$$\eta_t + c_o\eta_x + \alpha H(\eta_{xx}) = \beta^2\left[\eta_\tau + \alpha H\left(\eta_{\xi\xi}\right)\right] = 0$$

So, I am not going into great depth here, but I am just highlighting the type of equations that I am getting and of course, where is the use of Hilbert transform in this particular example. So, I get the PDE in this long wavelength scenario as above.

So, well after changing this expression using my transformed variable, I get to this particular expression and this is said equal to 0. So, if I were to look at this particular equation in the transformed plane, this is the celebrated linear Benjamin Ono equation; so, linear Benjamin Ono equation.So, this is particularly useful to study deep water waves where again my $H(\eta)$ is defined as so, the way the second variable is t is defined as my Cauchy principal value integral as before.

$$H(\eta(x',t)) = \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{\eta(x',t)}{(x'-x)}dx'$$

So, this is my Benjamin Ono equation for the deep water case, where my variable η is the depth that we are studying; so, depth as a function of x and t the ocean depth ok. So, then moving on there is the non-linear version of let us say this is my expression V. So, there is a non-linear version of this equation V.



The non-linear Benjamin Ono; so, the non-linear Benjamin Ono equation is described by this following equation.

$$\eta_t + c_0\eta_x + c_1\eta\eta_x + \alpha H(\eta_{xx}) = 0 \qquad ...(6)$$

Notice that this is the introduction of this new term or the new non-linear term, that is an extra term which is present in this example.

And, the corresponding solution to this equation was studied analytically and found that the corresponding wave solution is as follows.

Solution:

$$\eta(x-ct) = \frac{a\lambda^2}{(x-ct)^2+\lambda^2}$$

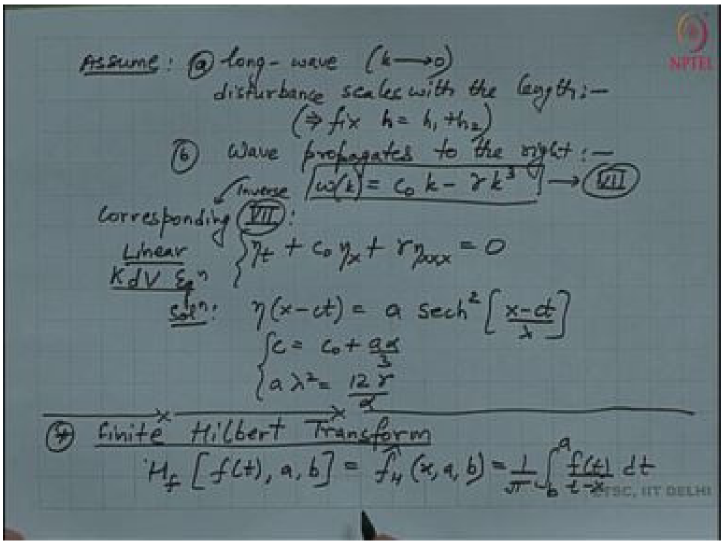
where,

$$c = c_0 + \tfrac{1}{2}ac_1$$
$$a\lambda = -\tfrac{4\alpha}{c_1}$$

$$c_0 = \sqrt{\left(\tfrac{\rho_2-\rho_1}{\rho_1}\right)gh}$$
$$\alpha = \left(\tfrac{\rho_2}{\rho_1}\right)\left(\tfrac{h_1c_0}{2}\right)$$

So, as I have discussed that these type of equations this let us say this is my expression (6) the expression the equation given by (5) and (6) is studied in a large variety of physical problems. So, starting the applications include lot of areas especially in quantum mechanics. So, in quantum mechanics in optics and people have been using these Benjamin Ono the linear and the non-linear version regularly in specially in these two areas that I have highlighted and also other allied areas.

Now, I am going to talk about another specific case of these deep and shallow water models. So, that is well the deep water model. So, s far I have introduced the deep water model the second case I wanted to say a little bit about was the shallow water model. So, shallow water theory again introduced by Benjamin that was in 1966.So, this theory was evolved slightly earlier. So, in this shallow water theory again the assumptions are as follows.



So, we assume that we have a long wavelength k going to 0 again the long wavelength assumption and the disturbance scales well the long wavelength disturbance scales with scales with the length ok. So, what it means is that I fix so, this particular assumption means that I fix my height given by $h_1 + h_2$ and take this limit k going to 0. So, further that is my first assumption further I assume that my wave propagates my wave propagates to the right ok.

My wave propagates to the right which means that my wave my dispersion relation is as follows. So, instead of an absolute value that was described in expression (3), I have this following expression:

$$\omega(k) = c_0 k - rk^3 \qquad ...(7)$$

So, I have taken I have got rid of the absolute value and taken a you know taken the cube. So, the absolute value is gone here. So, then what we have is that the corresponding. So, if I were to transform take an inverse transform of this expression.

So, let me call this as expression number 7. So, if I were to take the inverse transform of x in equation 7. So, the corresponding PDE is given by the famous linear PDE as follows.

$$\eta_t + c_0\eta_x + r\eta_{xx} = 0$$

So we have the following PDE, this is again a linear PDE and it is the linear caught away defreeze equation also called as the linear KdV equation ok. So, I am going to just give you the solution to this equation again with appropriate boundary conditions and initial condition the solution to this equation can be analytically found.

Solution:So, I can provide you with the solution well again eta is the depth of the water wave the solution is given in terms of a wave like solution as follows.

$$\eta(x-ct) = a\,\text{sech}^2\left[\frac{x-ct}{\lambda}\right]$$

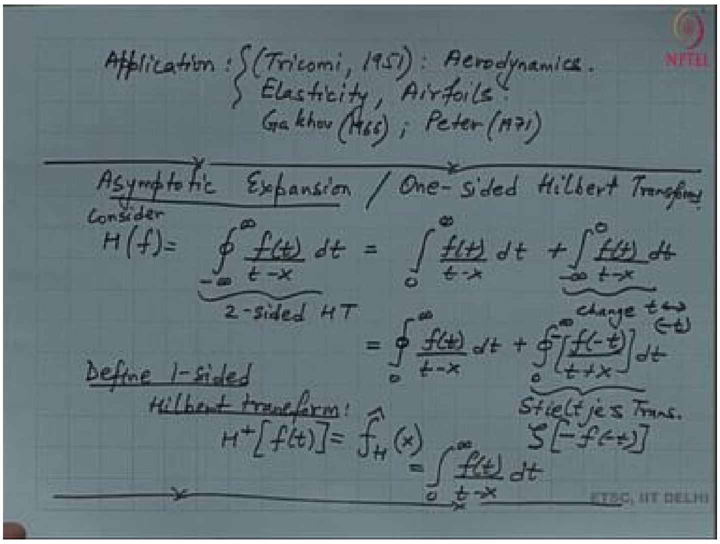
where,

$$\begin{aligned} c &= c_0 + \tfrac{a\alpha}{3} \\ a\lambda^2 &= \tfrac{12r}{\alpha} \end{aligned}$$

So, that completes more or less completes my discussion on water waves; however, I am also going to talk about again another application of Hilbert transform in relation to these wave equations and other areas of where we apply these PDEs especially aerodynamics and elasticity. So, then following in the same way it was shown that there are some applications of finite Hilbert transforms as well. So, finite Hilbert transform.

Finite Hilbert Transform:

$$H_f[f(t),a,b] = \hat{f}_H(x,a,b) = \frac{1}{\pi}\int_b^a \frac{f(t)}{t-x}dt$$



So, then so, well I am just going to highlight some of the applications the students are encouraged to see the applications in this areas, it was first introduced by a scientist known as Tricomi in this year, where he utilized these finite Hilbert transforms in his problems related to aerodynamics and then it was also shown in elasticity applications in elasticity of membranes, elasticity of rods and also in airfoil theory air foils by these people Ga khov there was a paper published in 1966 and by Peters published in 1971.

So, we see that without specifically solving some problems in Hilbert involving Hilbert transform, I have just highlighted some of the examples where Hilbert transforms are widely applicable. So, then I am going to talk about; I am going to talk about asymptotic expansions and most important I am also going to talk about one sided Hilbert transforms, one sided Hilbert transforms

$$H(f) = \oint_{-\infty}^{\infty}\frac{f(t)}{t-x}dt = \int_0^{\infty}\frac{f(t)}{t-x}dt + \int_{-\infty}^{0}\frac{f(t)}{t-x}dt$$

So, what I see that is this integral is well so, I call this as the standard Hilbert transform or the 2-sided Hilbert transform, 2-sided Hilbert transform.So, then I am going to break this integral at point $x = 0$ and of course, we have to take care of any arising singularities in the appropriate integrals that arise. So, breaking down into two integrals.

So, I am going to assume that my new set of two integrals are also principal value Cauchy principle value integrals, because we are not aware where is this x positive or is x negative.

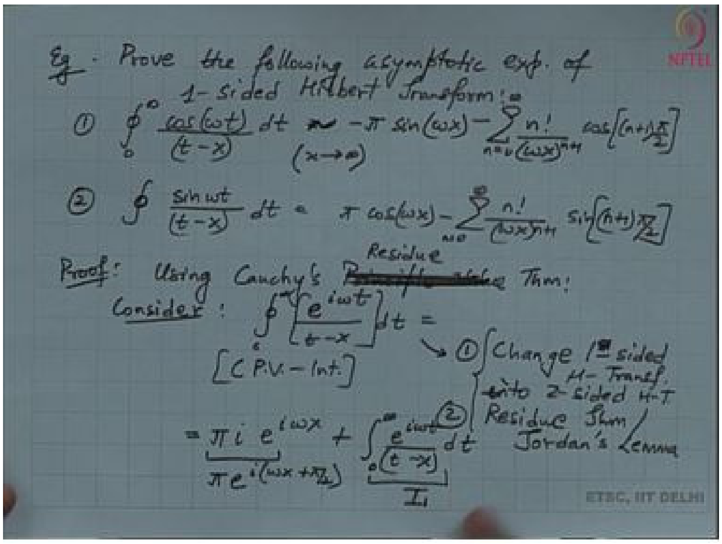
$$= \oint_0^\infty \frac{f(t)}{t-x} dt + \oint_0^\infty \left[\frac{f(-t)}{t+x} \right] dt$$

So, let me just absorb this minus sign inside the integral, and notice that this to begin with my 2-sided Hilbert transform is the sum of two 1-sided Hilbert transform. Later on I am going to show that this is another special integral transform known as the Stieltjes transform, the Stieltjes transform or this is denoted by this curly S of the function negative f of negative t. This particular in Stieltjes is transform is a 1-sided Hilbert transform.

So then let me just say that conclude this part of the discussion by saying that my 1-sided Hilbert transform my 1-sided Hilbert transform is defined as

$$H^+[f(t)] = \hat{f}_H(x) == \int_0^\infty \frac{f(t)}{t-x} dt$$

So then let us see some of the properties of this 1-sided Hilbert transform. So, in particular I have some result related to the asymptotic expansion of this Hilbert transform.



So, the problem says prove the following expression. So, prove the following asymptotic expansion of 1-sided Hilbert transform ok. So, the expansion result is as follows.

1.
$$\oint_0^\infty \frac{\cos(\omega t)}{(t-x)} dt \sim -\pi \sin(\omega x) - \sum_{n=0}^{\infty} \frac{n!}{(\omega x)^{n+1}} \cos\left[(n+1)\frac{\pi}{2}\right]$$

2.
$$\oint \frac{\sin \omega t}{(t-x)} dt = \pi \cos(\omega x) - \sum_{n=0}^{\infty} \frac{n!}{(\omega x + 1)^{n+1}} \sin\left[(n+1)\frac{\pi}{2}\right]$$

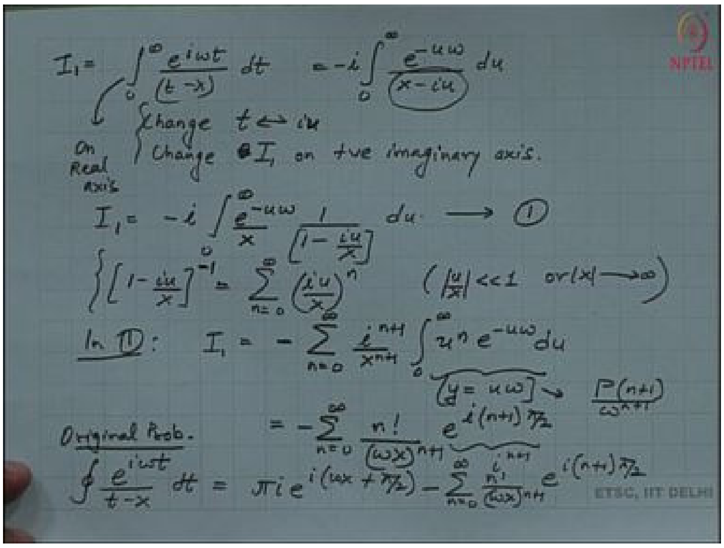
So, then the idea is to prove this result using my Cauchy's residue theorem right. So, what I mean to say was that I am going to use my Cauchy's residue theorem to give you the answer to this a proof. Notice that these two integrals are complementary in the sense that one is a Hilbert in one sided Hilbert transform of cosine and the other is the 1-sided Hilbert transform of sine.

Proof: Using Cauchy's residue theorem, So, the idea is I am going to first change this 1-sided Hilbert transform into 2-sided Hilbert transform. Consider,

$$\oint_0^\infty \left[\frac{e^{i\omega t}}{t-x}\right] dt == \pi i e^{i\omega x} + \int_0^\infty \frac{e^{i\omega t}}{(t-x)} dt$$

$$= \pi i e^{(i\omega x+\pi/2)} + \mathrm{I}_1$$

So, I have absorbed this iota inside this exponential using my Euler's formula. So, then let us look at this particular integral I call this as my integral I_1.



$$I_1 = \int_0^\infty \frac{e^{i\omega t}}{(t-x)} dt \quad = -i \int_0^\infty \frac{e^{-uw}}{(x-iu)} du$$

I am going to change make a change of variable. So, I am going to change my variable t as follows. So, introduce i u instead of a t.

So then so, what I have done is first make made a change of variables and also I have changed the integral from the negative to the positive. So, the earlier integral this integral was on the real axis right. So, I have further changed the integral on. So, change I 1 on the positive imaginary axis ok. So, imaginary axis ok so it was assumed that my original variable t was a real variable and then I have changed it to another variable which is purely on the positive imaginary axis ok. So, I have done this transformation. So, then let us look at the expression:

$$I_1 = -i \int_0^\infty \frac{e^{-uw}}{x} \left[\frac{1}{(1-\frac{iu}{x})} du\right] \qquad ....(1)$$

$$\left[1-\frac{iu}{x}\right]^{-1}=\sum_{n=0}^{\infty}\left(\frac{iu}{x}\right)^{n}$$

This is in the limit that ,

$$\left(\left|\frac{u}{x}\right| << 1 \quad \text{or } (|x| \to \infty)\right)$$

Now Equation (1):

$$I_1=-\sum_{n=0}^{\infty}\frac{i^{n+1}}{x^{n+1}}\int_0^{\infty}u^n e^{-uw}du$$
$$=-\sum_{n=0}^{\infty}\frac{n!}{(\omega x)^{n+1}}e^{i(n+1)\pi/2}$$

So, then I am already there in my result I see that my. So, my I_1 I leave this expression as it is and I come back to my original problem, my original problem was I am trying to evaluate this particular integral.

$$\oint\frac{e^{iwt}}{t-x}dt=\pi i e^{i(\omega x+\pi/2)}-\sum_{n=0}^{\infty}\frac{n!}{(\omega x)^{n+1}}e^{i(n+1)\pi/2}$$

We see that this particular sum, we now take the real and the imaginary part of this sum and equated to the corresponding real and the imaginary part of the left hand side and we are going to arrive at our expressions are both the expressions.