Integral Transforms and Their Applications
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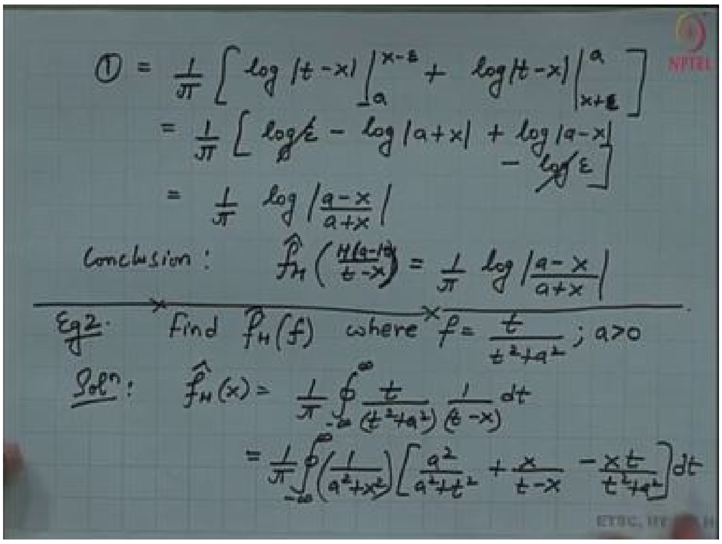
Lecture – 11
Introduction to Hilbert Transforms Part - 02

Example 2:find the Hilbert transform of a function

$$\hat{f}_H(f) \text{ where } f = \frac{t}{t^2+a^2}; a > 0$$



Solution:

$$\hat{f}_H(x) = \frac{1}{\pi}\oint_{-\infty}^{\infty} \frac{1}{t^2+a^2}\frac{1}{(t^2-x)}dt$$

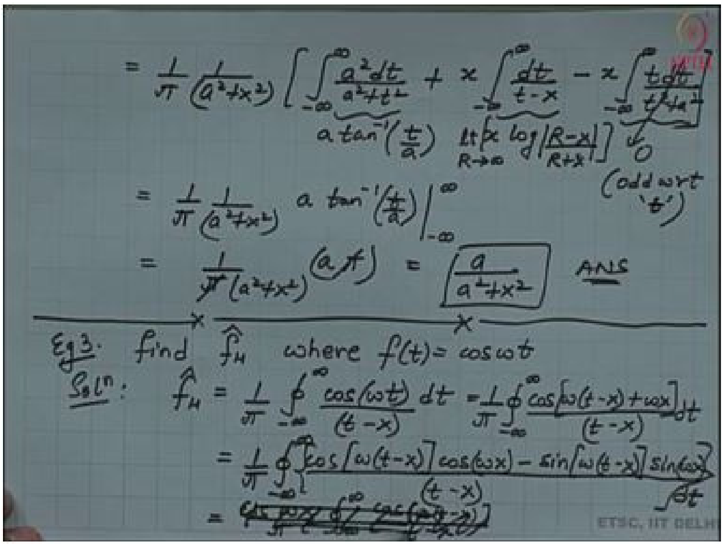
$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{a^2+x^2}\right)\left[\frac{a^2}{a^2+t^2} + \frac{x}{t-x} - \frac{xt}{t^2+a^2}\right]dt$$

$$= \frac{1}{\pi}\frac{1}{(a^2+x^2)}\left[\int_{-\infty}^{\infty}\frac{a^2 dt}{a^2+t^2} + x\int_{-\infty}^{\infty}\frac{dt}{t-x} - x\int_{-\infty}^{\infty}\frac{t}{t^2+a^2}dt\right]$$

$$= \frac{1}{\pi}\frac{1}{(a^2+x^2)} a\tan^{-1}\left(\frac{t}{a}\right)\Bigg|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi(a^2+x^2)}(a\pi)$$

$$= \frac{a}{a^2+x^2}$$



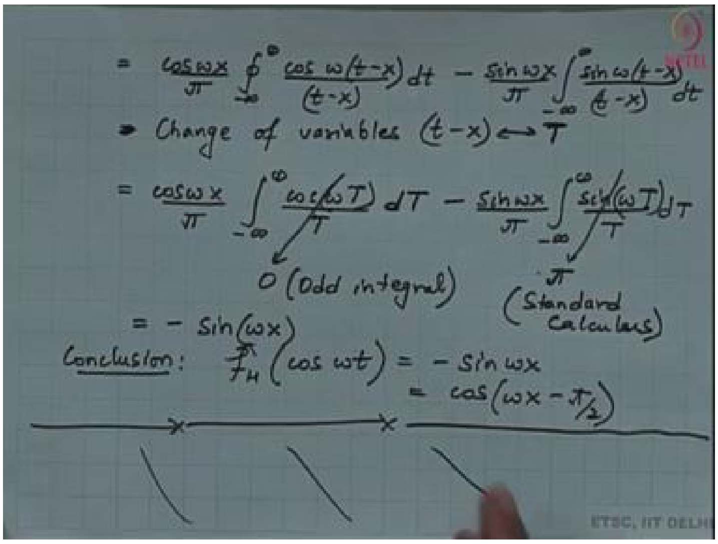
Example 3:

$$\text{Find } \hat{f}_H \text{ where } f(t) = \cos\omega t \qquad .$$

Solution:

$$\hat{f}_H = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega t)}{(t-x)} dt = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega(t-x)+\omega x)}{(t-x)} dt$$

$$= \frac{1}{\pi}\oint_{-\infty}^{\infty} \left[\frac{\cos(\omega(t-x))]\cos(\omega x) - \sin[\omega t - x]sin(\omega x)}{t-x}\right] dt$$



$$= \frac{\cos\omega x}{\pi}\oint_{-\infty}^{\infty} \frac{\cos\omega(t-x)}{(t-x)} dt - \frac{\sin\omega x}{\pi}\int_{-\infty}^{\infty} \frac{\sin\omega(t-x)}{(t-x)} dt$$

So, let me use a change of variables change of variables. So, I change my variable $t-x$ to the variable T, let us say another dummy variable here. So, we see that:

$$= \frac{\cos\omega x}{\pi}\int_{-\infty}^{\infty} \frac{\cos(\omega T)}{T} dT - \frac{\sin\omega x}{\pi}\int_{-\infty}^{\infty} \frac{sin(\omega T)}{T} dT$$

So, next we see that this first integral is an odd integral. The term on the numerator is even, the term in the denominator is odd. So, for an odd integral I get a 0 ok. For this particular integral the value is known from Standard Calculus textbooks. So, I am going to right away evaluate the value of this integral to give you the answer and its equal to π.

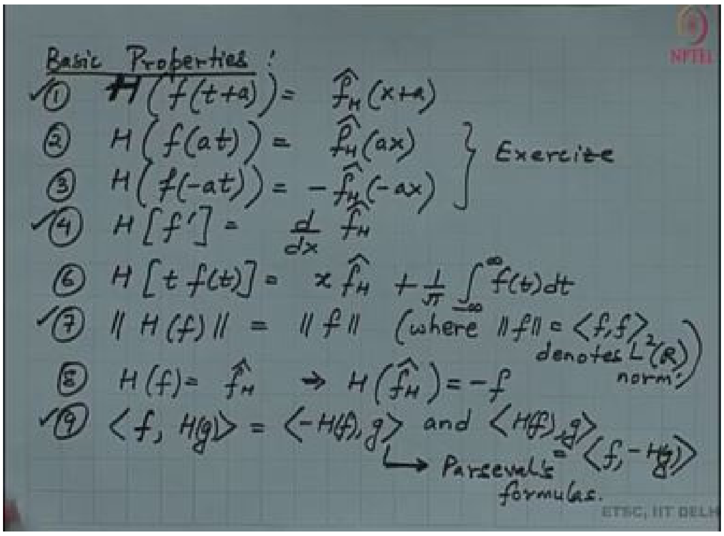
So, please refer to Standard Calculus to see that the value of this infinity integral is π.

So, I get that my answer is well the first integral is 0, the second integral gives me a π and I get my answer to be

$$= -\sin(\omega x)$$

$$\text{Conclusion: } \hat{f}_H(\cos\omega t) = -\sin\omega x = \cos(\omega x - \pi/2)$$

I want to highlight here is that that the Hilbert transform does a phase shift of an angle of 90 degrees on the Fourier variable, the Fourier transform variable ok. So, then I have lots of properties of Hilbert transforms which I want to write it down one by one and I am going to prove some of those results ok.



Basic Properties:

$$(1).\quad H(f(t+a)) = \hat{f}_H(x+a)$$

$$(2).\quad H(f(at)) = \hat{f}_H(ax)$$

$$(3).(f(-at)) = -\hat{f}_H(-ax)$$

$$(4).\,[f'] = \frac{d}{dx}\hat{f}_H$$

$$(5).[tf(t)] = x\hat{f}_H + \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)dt$$

$$(6). \quad \|H(f)\| = \|f\|$$

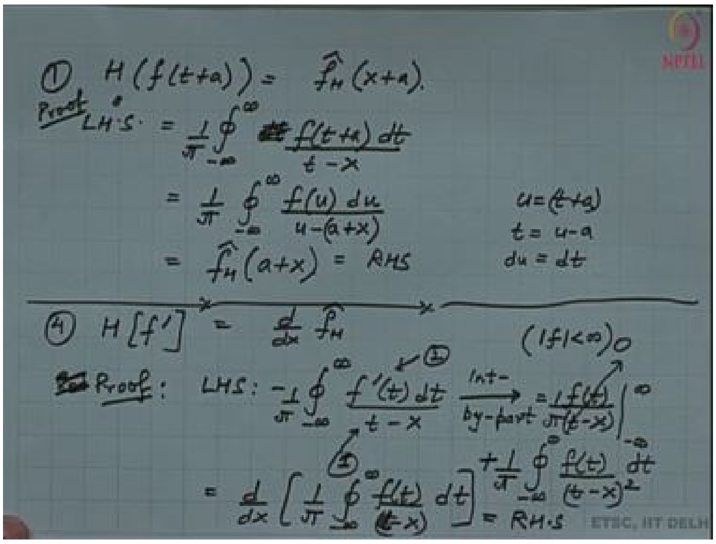
So, it denotes an inner product in the square integrable function sense; so, square integrable over R ok.

$$(7).(f) = \hat{f}_H \quad \Rightarrow H\left(\hat{f}_H\right) = -f$$

$$(8). \quad \langle f, H(g)\rangle = \langle -H(f), g\rangle \tag{1}$$

$$\langle H(f), g\rangle = \langle f, -H(g)\rangle \tag{2}$$

I am going to denote these inner products results by the Parceval's formulas. So, these are the Parceval's formulas for in relation to the Hilbert transform. So, I am going to show you some results here. So, out of all these nine results let me show you the proof of the first one and the proof of the second third follows in a similar fashion as first. I am going to show you the proof of the fourth one and I am going to use well I can also show right now I can show the proof of the seventh one and also let say well the eighth one I have already shown and I can show the ninth one as well right away. For the others I leave it as exercise. So, students should definitely try this at home to derive all these properties. So, let me show you some of the selected ones how we get to the result.



Proof 1:

$$H(f(t+a)) = \widehat{f}_H(x+a) \tag{3}$$

$$L \cdot H \cdot S = \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(t+a)dt}{t-x}$$

$$= \frac{1}{\pi} \oint_{-\infty}^{\infty} \frac{f(u)du}{u-(a+x)}$$

So, I substitute instead of $t+a$ I substitute u or instead of t, I substitute $u-a$ and also that $du = dt$.

$$= \hat{f}_H(a + x)$$

So, then let us look at the fourth case which I wanted to show. So, I have to show that the Hilbert transform of the derivative is the derivative of the transform function.

Proof 4:

$$H\left[f'\right] = \frac{d}{dx}\hat{f}_H$$

$$\text{LHS:} \quad -\frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{f'(t)dt}{t-x} = \frac{1}{\pi}\frac{f(t)}{\sin(t-x)}\bigg|_{-\infty}^{\infty} + \frac{1}{\pi}\oint_{-\infty}^{\infty}\frac{f(t)}{(t-x)^2}dt$$

$$= \frac{d}{dx}\left(\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{(t-x)}dt\right)$$