Integral Transforms and Their Applications
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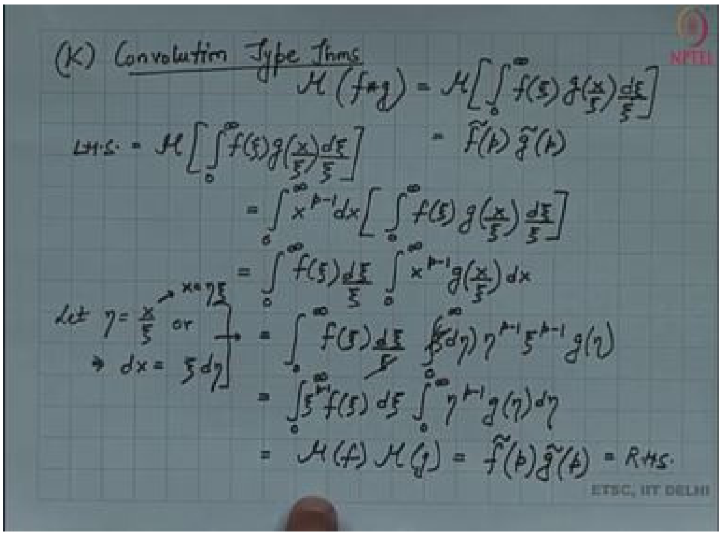
Lecture – 10
Introduction to Mellin Transforms Part - 03

So, then in the next slide I am going to highlight another result of Mellin transforms namely, the convolution of the Mellin transform of the convolution of two functions. So, let me write what the result is. So, I am going to show you a convolution type of a result for the Mellin transform of the convolution of two functions.



(K). Convolution Type Theorem:

$$\mathcal{M}\left(f^{*}g\right) = \mathcal{M}\left[\int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right]$$
$$= \widetilde{f}(p)\widetilde{g}(p)$$

$$LHS := \mathcal{M}\left[\int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right]$$

$$= \int_0^\infty x^{p-1} dx \left[\int_0^\infty f(\xi) g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}\right]$$

$$= \int_0^\infty f(\xi)\frac{d\xi}{\xi}\int_0^\infty x^{p-1} g\left(\frac{x}{\xi}\right) dx$$

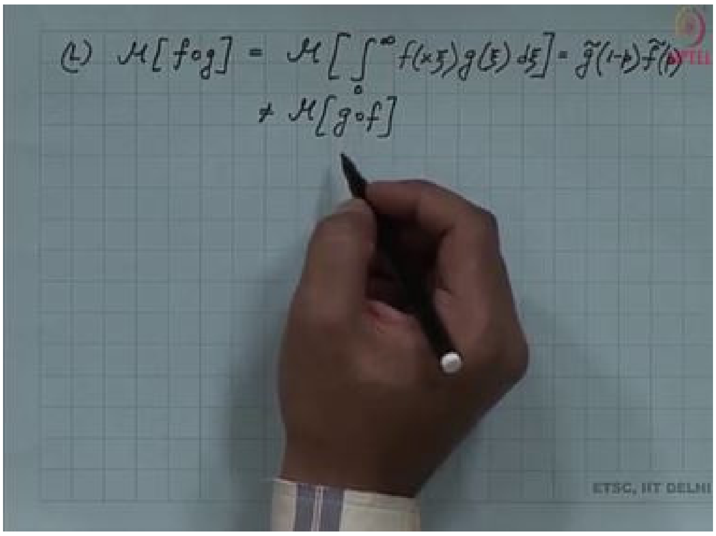
Let $\eta = \frac{x}{\xi}$ or $x = \eta\xi$
$\Rightarrow dx = \xi d\eta$

$$= \int_0^\infty f(\xi)\frac{d\xi}{\xi}\int_0^\infty \xi d\eta\Big)\, \eta^{p-1}\xi^{p-1} g(\eta)$$

$$= \int_0^\infty \xi^{p-1} f(\xi) d\xi \int_0^\infty \eta^{p-1} g(\eta) d\eta$$

$$= \mathcal{M}(f)\mathcal{M}(g) = \widetilde{f}(p)\widetilde{g}(p) = R.H.S$$

This is also the same notation, that is the product of 2 Mellin transforms and we get that this is what our right hand side is for our result ok. So, then there is another result that I have to show.



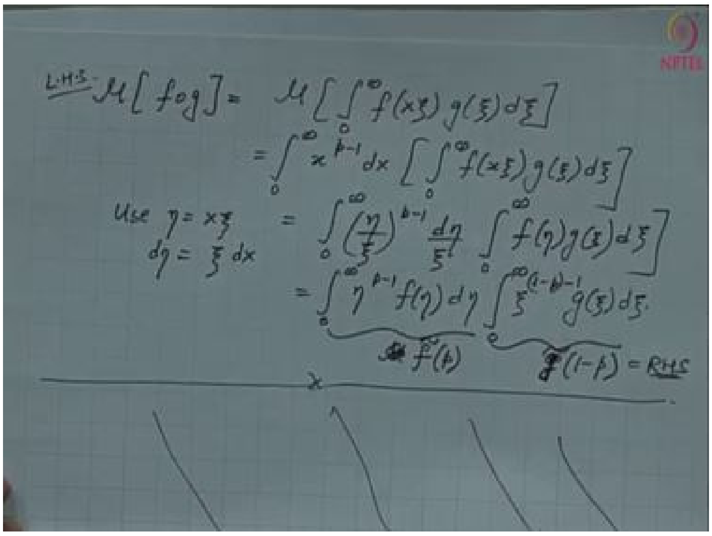
So, I see that the next result that we have is I have to show that what happens to the Mellin transform of the composition of the 2 functions ok.

(L).

$$\mathcal{M}[f \circ g] = \mathcal{M}\left[\int_0^\infty f(x\xi) g(\xi) d\xi\right] = \widetilde{g}(1-p)\widetilde{f}(p)$$

$$\neq \mathcal{M}[g \circ f]$$

I see that well before we move ahead, I just want to say that this is certainly not equal to the transform the Mellin transform of the composition in the reversed sense. So, this is because in this particular case, the product will have arguments which will be reversed. So, we will see that later on that, the Mellin transform of this particular composition is not equal to the transform of the composition which is reversed ok. What I have is that the Mellin transform of f composition with g is given by the Mellin transform.



$$L.H.S = \mathcal{M}[f \circ g] = \mathcal{M}\left[\int_0^\infty f(x\xi)g(\xi)d\xi\right]$$

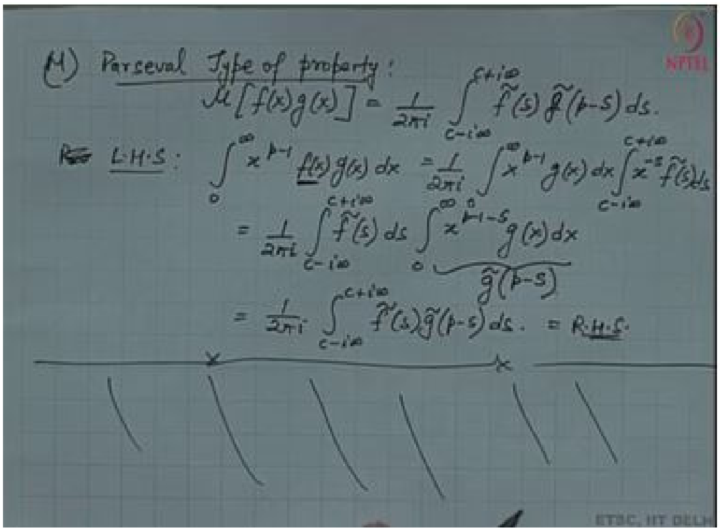
$$= \int_0^\infty x^{p-1}dx\left[\int_0^\infty f(x\xi)g(\xi)d\xi\right]$$

$$\text{use,}\quad \eta = x\xi$$
$$\Rightarrow d\eta = \xi dx$$

$$= \int_0^\infty \left(\frac{\eta}{\xi}\right)^{p-1}\frac{d\eta}{\xi}\int_0^\infty f(\eta)g(\xi)d\xi$$

$$= \int_0^\infty \eta^{p-1}f(\eta)d\eta\int_0^\infty \xi^{(1-p)-1}g(\xi)d\xi$$

$$= \widetilde{g}(1-p)\widetilde{f}(p) = R.H.S$$



(M) Parseval's Type of Property:

$$\mathcal{M}[f(x)g(x)] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\tilde{g}(p-s)ds$$

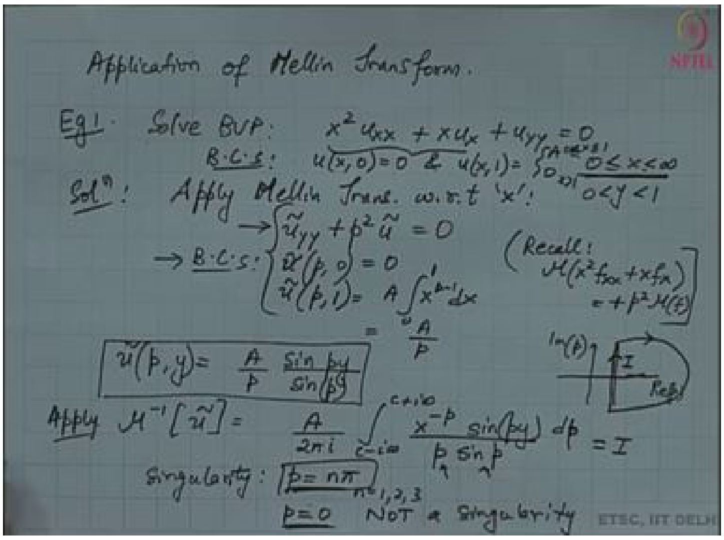
$$\text{LHS: } \int_0^\infty x^{p-1}f(x)g(x)dx$$

$$= \frac{1}{2\pi i}\int_0^\infty x^{p-1}g(x)dx \int_{c-i\infty}^{c+i\infty} x^{-s}\tilde{f(s)}ds$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{f}(s)ds \int_0^\infty x^{p-1-s}g(x)dx$$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{f}(s)\widetilde{g}(p-s)ds = R.H.S$$

So, we have seen, lots of properties of Mellin transform application of Mellin transform.



Application of Mellin Transform:
Example(1):Solve Boundary Value Problem:

$$x^2u_{xx} + xu_x + u_{yy} = 0$$

B.C.S:

$$\begin{aligned} u(x,0) = 0 \quad \& \quad u(x,1) = A; &\quad 0 \le x \le \infty \\ 0; &\quad 0 < y < 1 \end{aligned}$$

Solution:, I am going to apply Mellin transform with respect to the variable x here:
Now B.C.S is,

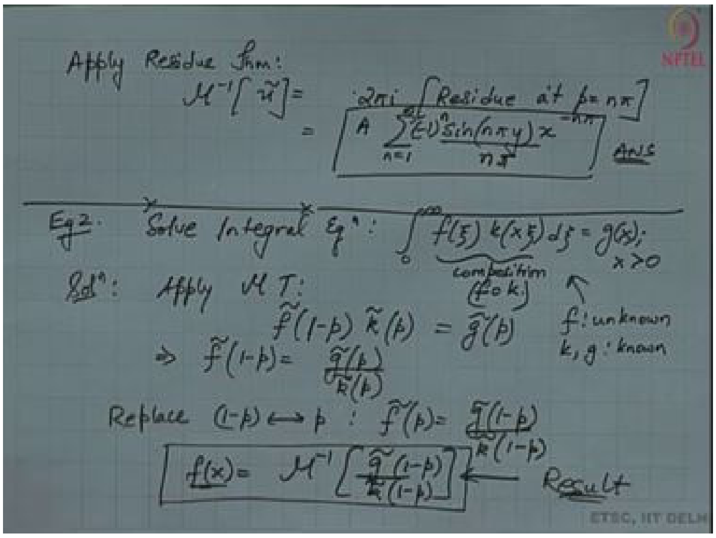
$$\begin{aligned} \rightarrow u_{yy} + p^2\tilde{u} &= 0 \\ \widetilde{u}(p,0) &= 0 \\ \widetilde{u}(p,1) &= A\textstyle\int x^{p-1}dx \\ &= \tfrac{A}{p} \end{aligned}$$

$$\widetilde{u}(p,y) = \frac{A}{p}\frac{\sin py}{\sin(p)}$$

$$\text{Apply } \mathcal{M}^{-1}[\tilde{u}] = \frac{A}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{-p}\sin(py)}{p\sin p}dp = I$$

$$\text{singularity: } p = n\pi \quad n = 1,2,3...$$

$$p = 0 \quad \text{:not a singularity}$$



Apply Residue Theorem:

$$\mathcal{M}^{-1}(\widetilde{u}) = 2\pi i[\text{Residue at p} = n\pi]$$

$$= A\sum_{n=1}^{\infty}[-1]^n \frac{\sin(n\pi y)x^{-n\pi}}{n\pi}\Bigg]$$

Example(2):Solve Integral Equation:

$$\int_0^{\infty} \frac{f(\xi)^k(x\xi)d\xi = g(x); x>0}{\text{composition}(f\circ k)}$$

Solution:Apply Mellin Transform:

$$\tilde{f}(1-p)\tilde{k}(p) = \widehat{g}(p)$$

$$\Rightarrow \widetilde{f}(1-p) = \frac{\tilde{q}(p)}{\tilde{k}(p)}$$

$$\text{Replace } (1-p) \leftrightarrow p:$$

$$\tilde{f}(p) = \frac{\tilde{g}(1-p)}{k(1-p)}$$

$$f(x) = \mathcal{M}^{-1}\left[\frac{\tilde{g}(1-p)}{\tilde{k}(1-p)}\right]$$