Integral Transform and Their Applications

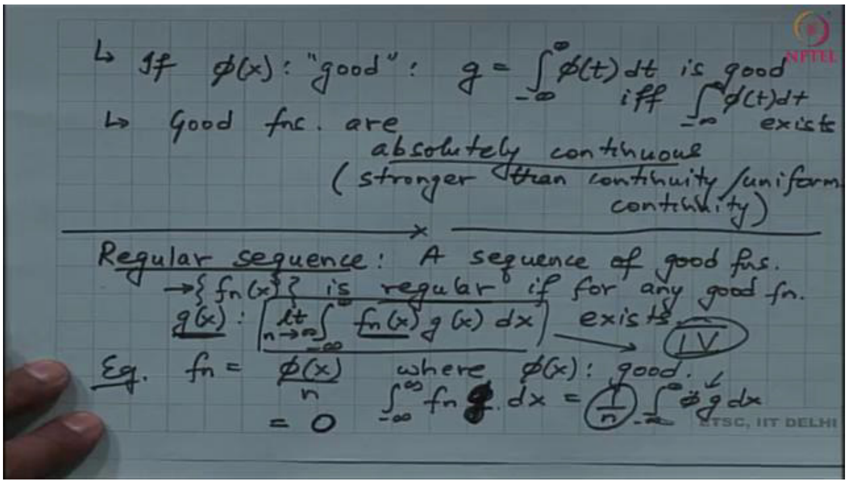
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute Of Information Technology

Lecture-01
Introduction to Fourier Transforms Part 3

$$\text{If } \phi(x) : \text{"}good\text{"} : g = \int_{-\infty}^{\infty} \phi(t)dt \text{ is good iff } \int_{-\infty}^{\infty} \phi(t)dt \text{ exists}$$

good functions are absolutely continuous (stronger than continuity/ uniform continuity)
<u>Regular Sequences</u> : A sequence of good functions $f_n(x)$ is regular if for any good function $g(x) : \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)g(x)dx$ exists.
Eg:

$$f_n = \frac{\phi(x)}{n} \text{ where } \phi(x) : good : \int_{-\infty}^{\infty} f_n dx = \frac{1}{n}\int_{-\infty}^{\infty} \phi g dx = 0$$



Well let us you know move on to the next module, I am going to define another notion called regular sequence ok. So, what do I mean by regular sequence? A sequence a sequence of good functions let us say I denote by the sequence as $f_n(x)$, a sequence of good function is regular if for any good functions $g(x)$ I have the following I have the following integral $\int f_n(x)g(x)dx$ this exists right.
So, if I can integrate with respect to this good function $g(x)$, if I can integrate this sequence of functions $f_n(x)$ and they exist then this sequence is called the sequence is called regular functions right well. To give you an example if I have this function f_n given by $\phi_n(x)$ right where my $\phi(x)$ is a good function right. So, then what do I get? This integral of $\int_{-\infty}^{\infty} f_n g(x)dx$ well in this case g's are phi right well let us say for a general g.

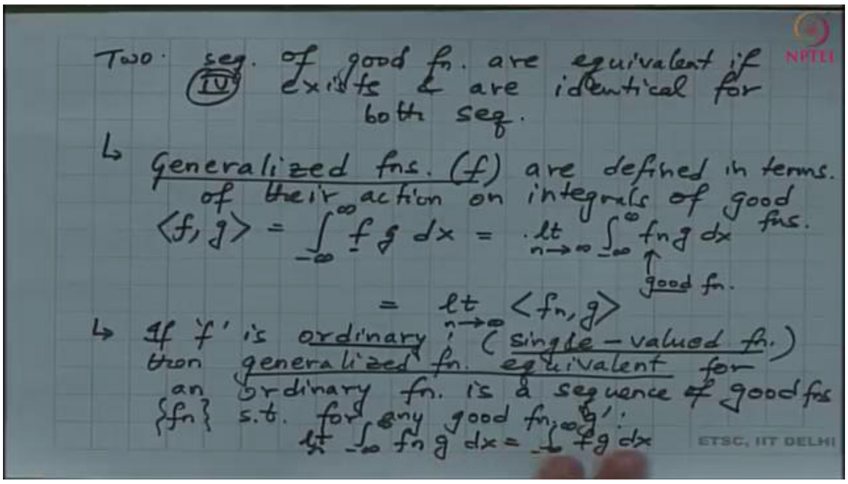
So, $f_n(g)dx$, I get this equal to $\frac{1}{n}\int_{-\infty}^{\infty} \phi g dx$ right. So, this is a good function integral of two good functions, integral of the product of two good functions will also exists. And then as well I need to mention that this integral should exist in the limit and tending to infinity right. So, what happens in this limit? In this limit, I get for this example as I take the limit the integral $\to 0$. So, as my $n \to \infty$, the integral $\to 0$ because the numerator in the numerator the integral exists right.

Because, its a product of two good functions ok. So, then I need to define some equivalence; let me call that let me call this expression as my expression IV right. So, I am going to call this expression as expression IV.

Two sequences of good functions are equivalent if IV exists and are identical for both sequences.
<u>Generalized function (f)</u> are defined in terms of their action on integral of good functions.

$$< f, g >= \int_{-\infty}^{\infty} fgdx = \lim_{n\to\infty} \int_{-\infty}^{\infty} f_n gdx = \lim_{n\to\infty} < f_n, g >$$

If 'f' is ordinary (single valued function)then generalized function equivalent for an ordinary fn. is a sequence of good fn. f_n st. for any good function 'g':
$\lim_n \int_{-\infty}^{\infty} f_n gdx = \int_{-\infty}^{\infty} fgdx$



So, two sequences two sequences of good functions are equivalent if my IV exists and are identical for both sequence right. So, when I am going to compare two sequence of regular functions right well two sequences of good function, I am going to compare their corresponding integral which was defined by IV ok. Then what I have is that my generalized function. So, that is the thing that I wanted to define to begin with. My generalized functions let us denote it by f my generalized functions are defined in terms of their action on integrals of good functions right.

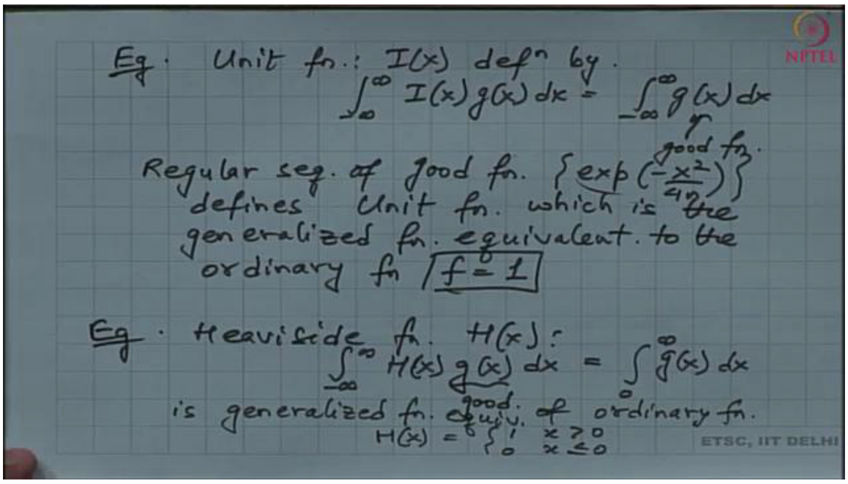
So, what do I mean by that? If I have this function f and I am using this bracket notation and this is defined as follows. So, this (f, g) is actually $\int_{-\infty}^{\infty} fgdx$ right. Now since I have this generalized function. So, the generalized function is also its a regular sequence of good function. So, I am going to denote replace my f as $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n g(x)dx$

So, these are my sequences of good functions right. So, I get that this is also equal to $\lim_{n\to\infty}(f_n, g)$. So, my action of f on g is defined in the limiting sense of f_n on g right. So, this is how I define my generalized function ok. So, then finally, I want to introduce the so, called ordinary function. So, if f if my function f is ordinary, it is also; when I say ordinary functions, ordinary functions are your regular functions well ordinary functions are your normal the functions that you have you know or the functions which are so, called single valued functions, functions that we are all aware in calculus.

We I am going to denote it by ordinary function right. So, if a function is ordinary then what I have is that the generalized the generalized function equivalent. Let me just highlight all these notations and I am going to tell you why all these notations are being introduced. So, the generalized function equivalent for an ordinary function generalized function equivalent for an ordinary function is a sequence of good functions is a sequence of good functions $f_n(x)$ such that for any good function g, I have the following integral true. So, I have that $\lim_n \int_{-\infty}^{\infty} f_n g(x)dx = \int_{-\infty}^{\infty} fgdx$.

So, whenever I want to highlight. So, if I have an ordinary function. So, notice where these definition make sense. So, if I have an ordinary function and I am not able to find the Fourier transform of that ordinary function, then what I do is I represent that ordinary function into this generalized function equivalent using this following sequence and I am able to find the Fourier transform of that generalized function. And, then I take the limit n tending to infinity and that will give me the Fourier transform of that ordinary function in the generalized sense ok. So, I will show you with an example. So, let me just give you one quick example.

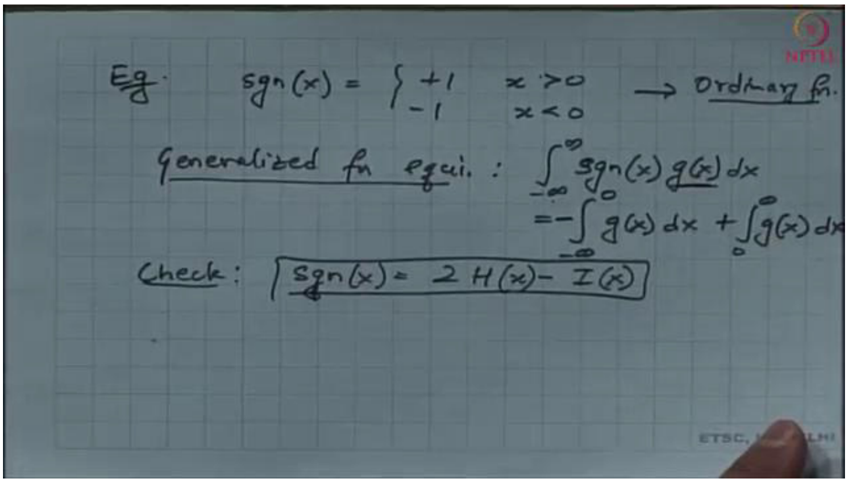
Eg. Unit function : I(x) defined by :
$\int_{-\infty}^{\infty} I(x)g(x)dx = \int_{-\infty}^{\infty} g(x)dx$
Regular sequnce of good function $\exp(\frac{-x^2}{4n})$ defines unit function which is the generalized function equivalent to the ordinary fn. $\mathbf{f = 1}$
Eg. Heaviside function $H(x)$: $\int_{-\infty}^{\infty} H(x)g(x)dx = \int_{0}^{\infty} g(x)dx$
is generalized fn. equivalent of ordinary fn.
$\text{H(x)} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{otherwise} x \leq 0. \end{cases}$



So, if I have a unit function a unit function I denote it by $I(x)$ right and it is defined by defined by$\int_{-\infty}^{\infty} I_x g(x)dx = \int_{-\infty}^{\infty} g(x)dx$. So, it is an identity function and its notation is that is just like multiplication with a one; so, where g is a good function right. So, then what I have is that my regular sequence consider this regular sequence; my regular sequence of good functions which is given by $e^{-(x^2 4n)}$ right. Well it defines this regular sequence, it defines my unit function which is the generalized function equivalent to the ordinary function ordinary function $f = 1$.
So, here I know what is the; I am easily able to find the Fourier transform of f equal to one; however, suppose if I had a function for which I was not able to find Fourier transform, then I the idea is to use the corresponding sequence of good functions here in this case it is this one, and I am able to find the Fourier transform of this sequence and take the limit n tending to infinity in that integral. So, that is going to give me the Fourier transform of the ordinary function corresponding ordinary function. So, similarly another example is if I have the Heaviside function.
So, H(x) is defined by this following generalized function equivalent. H(x) g(x) again without lose of generality these are good functions right. This is given to be as the definition says $\int_{0}^{\infty} g(x)dx$ again using my Heaviside function. So, this is the generalized function equivalent generalized function equivalent of the ordinary function H(x) given by 1 if x is positive and 0 if x is non positive ok. Let us move a while.
Eg. sgn(x) =
$\begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{otherwise} x < 0. \end{cases}$ <u>Generalized fn. equi.</u>:

$$\int_{-\infty}^{\infty} sgn(x)g(x)dx$$
$$= -\int_{-\infty}^{0} g(x)dx + \int_{0}^{\infty} g(x)dx$$

$Sgn(x) = 2H(x) - I(x)$



So, well there is one more example which is of our importance that is the so, called sign function. So, what is sign function? So, the sign function is defined to be

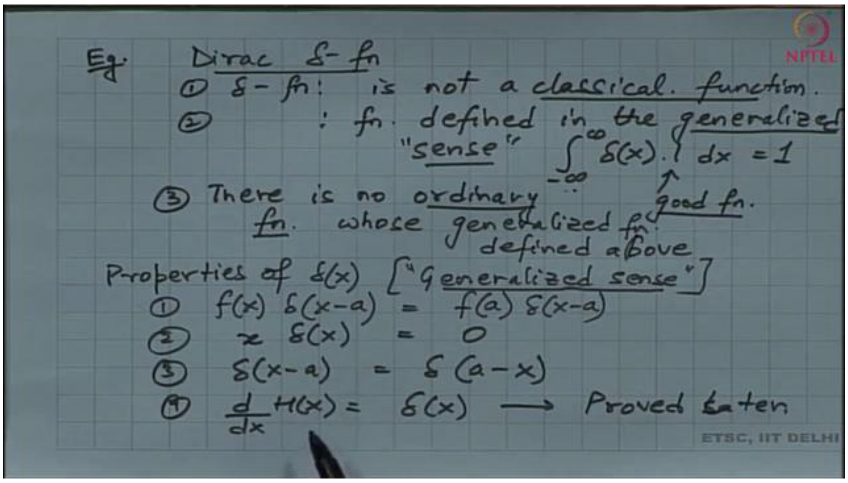
$$\begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{otherwise} x < 0. \end{cases}$$

So, this is my ordinary function this is the ordinary function that I am talking about sign function. The generalized function equivalent will be $\int_{-\infty}^{\infty} sgn(x)g(x)dx$ which is again we replace what is sgn(x) ? So, from $-\infty$ to 0 sign of x is minus 1.

So, I have a -g(x)dx plus from 0 to infinity I have a plus sign. So, this becomes $+g(x)dx$ right. So, for this ordinary function sign x the generalized function equivalent definition is as follows. So, it is the action on this good function g(x) right also please check please check this useful relation that sign of x is actually equal to 2 times Heaviside function minus my unit function I am going to use this relation quite frequent ok. So, why I have introduced all these notion of good function generalized function is to go through the example that I am going to describe next ok.

Eg. Dirac δ- function

- $\delta - fn.$ is not a classical function
- fn. defined in the generalized "sense" $\int_{-\infty}^{\infty} \delta(x).1dx = 1$
- there is no ordinary function whose generalized function is defined above.
- $f(x)\delta(x-a) = f(a)\delta(x-a)$
- $x\delta(x) = 0$
- $\delta(x-a) = \delta(a-x)$
- $\frac{d}{dx}H(x) = \delta(x)$



The most important example that we have for which we do not have a proper definition of Fourier transform is the Dirac delta function. What is Dirac delta function? Well we all know most of us we know what is Dirac delta function. So, it turns out that, Dirac delta function is not a classical function although it is quite heavily used in physics chemistry biology applied mathematics everywhere, it is not a classical function right and well it is also a function well I can only define this function in the generalized sense. So, it can only be defined in the generalized sense why because, I know that $\int \delta(x) 1 dx = 1$. So, here my 1 is my good function right. So, in the generalized sense my delta function is a function can that can be defined right and also what we have is that there is there is no ordinary function there is no ordinary function whose generalized function is defined about right. So, what I need to say is that, my delta function is only defined well we all know what is delta function.

So, delta function is going to go to infinity at a particular point and 0 otherwise, but as such it is not a function in the classical sense, its only function in the generalized sense as defined by this integral. And there is no ordinary function whose generalized function equivalent is defined here ok. Now we all we are quite familiar with some of the properties of delta function. So, again let me just highlight some properties of delta function again these properties are defined in the generalized in the generalized sense ok.

So, the one of the property says that suppose you have a function let us say a good function times $\delta(x-a)$, that can also be given to be equal to $f(a) * \delta(x-a)$. Again you see this equality I have to look at this equality in the generalized sense, that is in that integral that I have defined here. Then another property that I have for delta function is $x * \delta(x)$ is 0, then I have $\delta(x-a)$ is an even function. So, $\delta(x-a)$ is $\delta(a-x)$ and finally, it turns out that the derivative of my Heaviside function is the delta function right. So, I am going to prove I am going to show it later soon ok. So, the properties these are some of the properties that are highlighted.

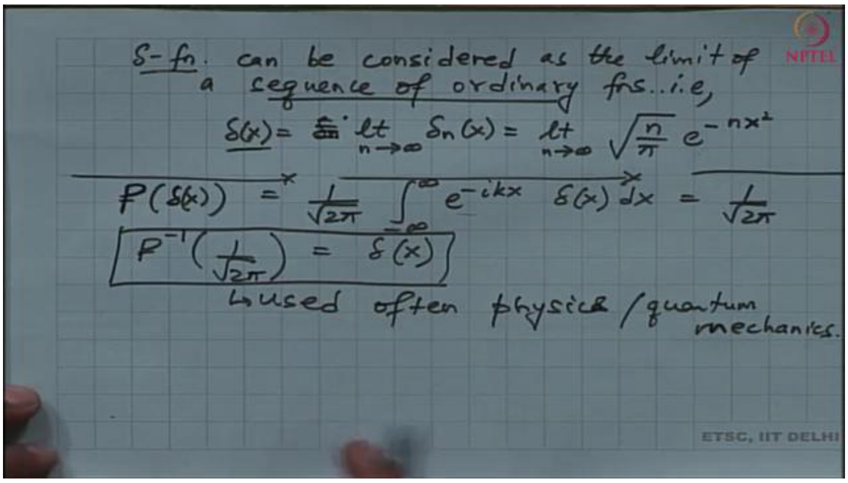
<u>δ-function</u> can be considered as the limit of a sequence of ordinary fns. i.e.,

$\delta(x) = \lim_{n\to\infty} \delta_n(x) = \lim_{n\to\infty} \sqrt{\frac{n}{\pi}} e^{-nx^2}$

$P(\delta(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \delta(x) dx = \frac{1}{\sqrt{2\pi}}$

$P^{-1}(\frac{1}{\sqrt{2\pi}}) = \delta(x)$ used often physics/ quantum mechanics

.

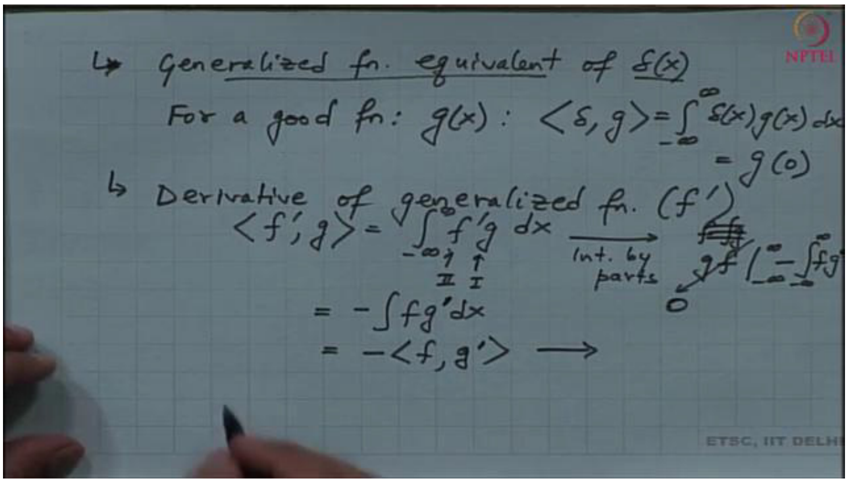


And, before I move ahead I want to mention that I can always approximate my delta function as a limit of the following ordinary functions. So, I can always approximate my delta function as a limit of this sequence what is this sequence? The sequence is $\delta(x)$ which is limit n tending to infinity $\delta_n(x)$ which is limit n tending to infinity. So, these individual functions delta n to be defined as: $\sqrt{\frac{n}{\pi}}e^{-nx^2}$. So, as my limit n goes to infinity, these functions delta n goes to my Dirac delta function. So, either I can represent my delta function as this sequence or I can represent my delta function in the generalized sense that is in terms of that integral representation ok. So, let me just show you some cases or some properties of delta function applied to Fourier transform. So, in particular if I want to evaluate the Fourier transform of delta function let us say Fourier transform of delta of x.
So, if I want to use the definition, this becomes $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx}\delta(x)dx$. Now from negative infinity to infinity I have. That there is only one point the delta function attains a non negative a non zero value and that is at equal to 0 and at x is equal to 0 this is 1 right. So, all I am left with is the factor $\frac{1}{2\pi}$ right. So, the integral is just 1 right. Conversely if I were to evaluate the Fourier inverse of $\frac{1}{2\pi}$ then that is my delta function ok. So, why I am denoting this is, because this particular definition is used often in physics or quantum mechanics. Definition of delta function as an inverse of Fourier transform ok. So, then let us move ahead.
Generalize fn. equivalent of $\delta(x)$
For a good fn. $g(x) :< \delta, g >= \int_{-\infty}^{\infty} \delta(x)g(x)dx = g(0)$
Derivative of generalized fn. (f')

$$< f', g >= \int_{-\infty}^{\infty} f'gdx$$

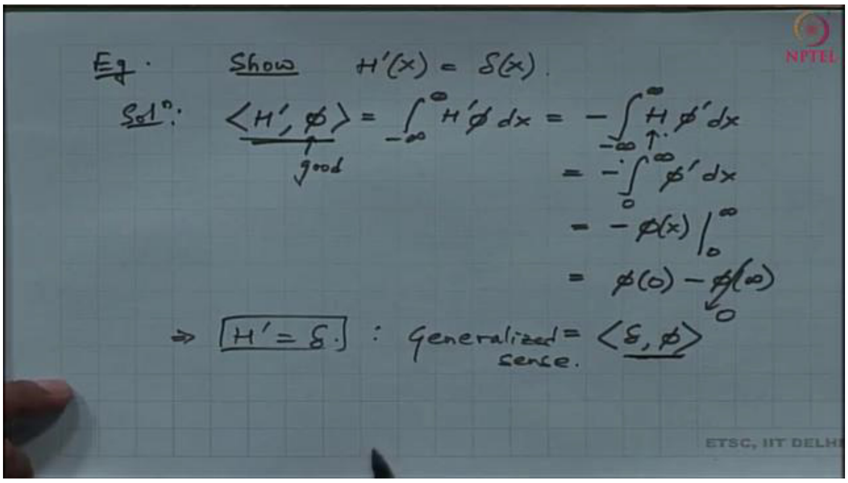
$$= -\int_{-\infty}^{\infty} fg'dx \qquad \text{[solving integral by parts]}$$

$$= - < f, g' >$$



Let me just give you bit more few more properties of generalized function. So, if I have generalized function equivalent, I am going to show you the generalized function equivalent of delta function right. So, which means I have to define it in terms of a good function. So, for a good function for a good function g(x), I have this bracket delta of g delta comma g which is defined to be $\int_{-\infty}^{\infty} \delta(x)g(x)dx$ this integral from which is nothing, but g at 0 right. So, that is my generalized function equivalent of delta function ok; then if I have to figure out the derivative of the generalized function derivative of the generalized function.
Let us say I denote it by f prime. So, then again using my bracket notation $f'g = \int_{-\infty}^{\infty} f'gdx$ right and I use my integration by parts. So, integration by parts then I am going to use this as my first function this as my second function. So, f integral g minus well I am going to use this as my first function this as my second function. So, I get g f evaluated at minus infinity to infinity, and then minus integral f g from minus infinity to infinity.
Now, this the fact that the functions are absolutely integrable because otherwise the Fourier transform would not have been defined, I get that these vanishes at the end point minus infinity and infinity because these are absolutely integrable function. So, I get that this is negative integral f g dx or this is also equal to f times g prime properly using the integration by parts. So, f times g prime, I get the bracket f comma g prime with the minus sign outside ok.
So, if I were to figure out the derivative of a generalized function it is negative times the bracket operator operating on the derivative of the good function why I am using this is, because I have the next example in mind. Eg. Show H'(x) = $\delta(x)$
soln.

$$\begin{aligned}
< H', \phi > &= \int_{-\infty}^{\infty} H'\phi dx \\
&- \int_{-\infty}^{\infty} H\phi' \\
&- \int_{0}^{\infty} \phi' \\
&- \phi(x)|_0^{\infty} \\
&= \phi(0) - \phi(\infty) \\
&= \phi(0) \\
\Rightarrow H' = \delta . Generalized\ sense &= < \delta, \phi >
\end{aligned}$$



So, if I have I want to show this example I want to show that the derivative of my Heaviside function is the delta function right. So, how to show that? Notice that if I were to use again the bracket notation. So, heavi derivative of the Heaviside function applied on a good function right.

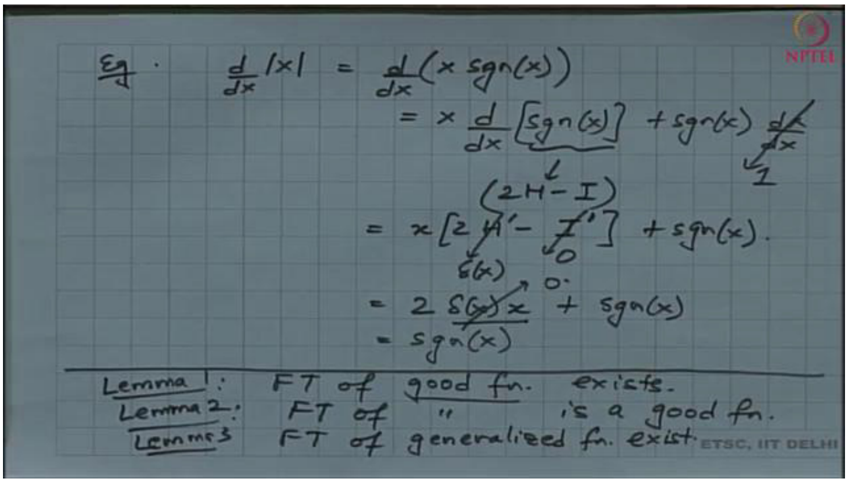
So, this is$\int_{-\infty}^{\infty} H'\phi dx$, and then again using my definition used earlier this is $\int_{-\infty}^{\infty} H\phi' dx$ right and of course, using my definition of Heaviside function, this becomes $-\int_0^{\infty} \phi' dx$. Now, this is derivative of phi and this is integral of phi. So, all get is a simple integration phi of x evaluated at with a minus sign evaluated from 0 to infinity right and then I get this is equal to phi at 0 well it vanishes at minus phi at infinity well this 1 vanishes because phi is a good function right.

So, I get well I know that phi of 0 is nothing, but the delta function operated on phi you can check that. So, I have that the bracket of H prime action on phi is delta action on phi; which means what I get is that h prime is delta right in the generalized in the generalized sense ok. Then there is one more example I need you to see.

Eg.

$$\begin{aligned}
\frac{d}{dx}|x| &= \frac{d}{dx}(xsgn(x)) \\
&= x\frac{d}{dx}[sgn(x)] + sgn(x)\frac{dx}{dx} \\
&= x[2H' - I'] + sgn(x) \\
&= 2\delta(x)x + sgn(x) \\
&= sgn(x)
\end{aligned}$$

Lemma 1. FT of good fn. exists.
Lemma 2. FT of good fn. is a good fn.
Lemma 3. FT of generalized fn. exist.



The derivative of the absolute value of x right this is also you can check that this is the derivative of x times sign of x right. You can check that this is indeed the case. So, we can possibly we can use our product rule here well with some caution product rule has to be used with some caution here. So, this will be equal to x times d dx sign of x plus sign of x dx dx right. So, then I this is 1 and you can check that my sign function is 2 times Heaviside function minus the unit function right. So, all I get this is x times 2 times Heaviside function derivative minus the unit function derivative plus sign of x right. You can check that this one goes to 0 please check that and this one is nothing, but the delta function right. So, I get this is $2\delta(x+sgn(x))$ 2 right and I know that $\delta(x)x \to 0$ by the property of the delta function. So, all I get that is that the derivative of $|x|\ is\ sgn(x)$ ok. Let me just wrap up this lecture by highlighting few of the lemmas that are useful. There is first lemma Fourier transform of good functions exists. Another lemma or result that is useful is, Fourier transform of good function is a good function right. Why I am showing you this lemmas is because to help you to evaluate Fourier transforms of ordinary functions where they are not where you are not able to evaluate them.

So, represent them in terms of good functions right and if I have there is one final result which tells us that if I have the Fourier transform of the good function, then the corresponding Fourier transform of the generalized function also exists right. So, if you are able show that the Fourier transform of the good function exists then the corresponding generalized function also exists.

So, thank you very much. So, I end this lecture at this point.