

Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi
Lecture – 10
Introduction to Mellin Transforms Part - 02

Next I am going to highlight some of the properties of Mellin transform,

The image shows handwritten notes on a lined notebook page. At the top left, it says "Properties of $\mathcal{M}(f)$ ". Below that, under the heading "(A) Scaling", it shows the formula $\mathcal{M}(f(ax)) = a^{-p} \tilde{f}(p)$ for $a > 0$. It then derives this from the definition of the Mellin transform $\mathcal{M}[f(ax)] = \int_0^\infty x^{p-1} f(ax) dx$, by substituting $s = ax$ and simplifying the integral. Under the heading "(B) Shifting", it shows the formula $\mathcal{M}[x^a f(x)] = \tilde{f}(p+a)$. It then derives this from the definition of the Mellin transform $\mathcal{M}[x^a f(x)] = \int_0^\infty x^{p+a-1} f(x) dx$, by substituting $s = x$ and simplifying the integral. The notes are written in blue ink on a light green background.

(A)Scaling:

$$\mathcal{M}(f(ax)) = a^{-p} \tilde{f}(p); \quad a > 0$$

$$\mathcal{M}[f(ax)] = \int_0^\infty x^{p-1} f(ax) dx$$

Substitute $s = ax$;

$$= \int_0^\infty \frac{s^{p-1} f(s)}{(a^{p-1}) a} ds$$

$$= a^{-p} \mathcal{M}(f) = \text{RHS}$$

(B).Shifting:

$$\mathcal{M}[x^a f(x)] = \tilde{f}(p + a)$$

Taking LHS;

$$\begin{aligned} \int_0^\infty x^{p-1} x^a f(x) dx &= \int_0^\infty x^{p+a-1} f(x) dx \\ &= \tilde{f}(p + a) \end{aligned}$$

SPPU

(C) $\mathcal{M}(f(x^a)) = \frac{1}{a} \tilde{f}\left(\frac{p}{a}\right)$: Exercise
 (D) $\mathcal{M}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] = \tilde{f}(1-p)$: "
 (E) $\mathcal{M}[(\log x)^n f(x)] = \frac{d^n}{dp^n} \tilde{f}(p)$
 Proof: $\frac{d(x^{k-1})}{dp} = (\log x)x^{k-1}$
 LHS: $\int_0^\infty (\log x)^n x^{k-1} f(x) dx$
 $= \int_0^\infty (\log x)^{n-1} (\log x) x^{k-1} f(x) dx$
 $= \int_0^\infty (\log x)^{n-1} \frac{d}{dp} x^{k-1} f(x) dx$
 $= \frac{d}{dp} \int_0^\infty (\log x)^{n-1} x^{k-1} f(x) dx$ - ETSC, IIT DELHI

(C).

$$\mathcal{M}(f(x^a)) = \frac{1}{a} \tilde{f}\left(\frac{p}{a}\right) \quad : \text{Exercise}$$

(D).

$$\mathcal{M}\left[\frac{1}{x} f\left(\frac{1}{x}\right)\right] = \tilde{f}(1-p) : \text{Exercise}$$

(E).

$$\mathcal{M}[(\log x)^n f(x)] = \frac{d^n}{dp^n} \tilde{f}(p)$$

Proof:

Hint: $\frac{d(x^{p-1})}{dp} = (\log x)x^{p-1}$

LHS:

$$\begin{aligned} & \int_0^\infty (\log x)^n x^{p-1} f(x) dx \\ &= \int_0^\infty (\log x)^{n-1} (\log x) x^{p-1} f(x) dx \\ &= \int_0^\infty (\log x)^{n-1} \frac{d}{dp} x^{p-1} f(x) dx \\ &= \frac{d}{dp} \int_0^\infty (\log x)^{n-1} x^{p-1} f(x) dx \dots \text{continuing inductively n times.} \end{aligned}$$

$$LHS : \frac{d^n}{dp^n} \int_0^\infty x^{p-1} f(x) dx$$

$$= \frac{d^n}{dp^n} \tilde{f}(p) = RHS$$

NITEE

-- continuing inductive ' n -times:

$$\begin{aligned} \text{LHS: } & \frac{d^n}{dp^n} \int_0^\infty x^{p-1} f(x) dx \\ &= \frac{d^n}{dp^n} \tilde{f}(p) = \text{RHS.} \end{aligned}$$

(F) Mellin Transforms of derivatives:

$$\mathcal{M}(f') = -(p-1) \tilde{f}(p-1) \quad \left[\begin{array}{l} \text{Assume} \\ x^{p-1} f(x) \xrightarrow{x \rightarrow 0} 0 \end{array} \right]$$

Proof: L.H.S.: $\mathcal{M}(f') = \int_0^\infty x^{p-1} f'(x) dx$

Int-by-parts $\int_0^\infty x^{p-1} f'(x) dx = \int_0^\infty x^{p-2} f(x) dx - \int_0^\infty (p-1)x^{p-2} f(x) dx$

$$\begin{aligned} &= -(p-1) \int_0^\infty x^{(p-1)-1} f(x) dx \\ &= -(p-1) \tilde{f}(p-1) = \text{R.H.S.} \end{aligned}$$

ETSC, IIT DELHI

So, this time I am going to highlight Mellin transforms of derivatives. So, Mellin transforms of derivatives are given by

(F). Mellin transforms of derivatives:

$$\mathcal{M}(f') = -(p-1) \tilde{f}(p-1)$$

Proof:

$$\text{L.H.S.: } \mathcal{M}(f') = \int_0^\infty x^{p-1} f'(x) dx$$

So, using integration by parts I get

$$\begin{aligned} & x^{p-1} f(x) \Big|_0^\infty - \int_0^\infty (p-1)x^{p-2} f(x) dx \\ &= -(p-1) \int_0^\infty x^{(p-1)-1} f(x) dx \\ &= -(p-1) \tilde{f}(p-1) = \text{RHS} \end{aligned}$$

(F) $\mathcal{M}(f'') = (p-1)(p-2) \tilde{f}(p-2)$ -- inductively

$$\begin{cases} \mathcal{M}(f^n) = (-1)^n (p-1) \dots (p-n) \tilde{f}(p-n) \\ = (-1)^n \frac{P(p)}{P(p-n)} \tilde{f}(p-n). \end{cases} \quad \left[\begin{array}{l} \text{provided} \\ \text{Assume } x^{p-1} f(x) \xrightarrow{x \rightarrow 0} 0 \\ r = 0, \dots, (p-1) \end{array} \right]$$

(G) $\mathcal{M}[x f'(x)] = -p \tilde{f}(p) \rightarrow \text{Exercise}$

(H) $\mathcal{M}[x^2 f''] = (-1)^2 p(p+1) \tilde{f}(p) \rightarrow "$

(I) Mellin Transform of diff. operators

$$\mathcal{M}\left[\left(\frac{d}{dx}\right)^2 f\right] = \mathcal{M}\left[\left(\frac{d}{dx}\right)\left(\frac{d}{dx}\right) f\right] = \mathcal{M}\left[x^2 f'' + x f'\right]$$

Used (H), (G) $= (-1)^2 p^2 \tilde{f}(p)$

ETSC, IIT DELHI

$$\mathcal{M}(f'') = (p-1)(p-2)\tilde{f}(p-2) \dots \text{inductively}$$

$$\mathcal{M}(f^n) = (-1)^n(p-1)\dots(p-n)\tilde{f}(p-n)$$

I assume that:

$$x^{p-1-r}f(x) \rightarrow 0 \quad \text{as, } x \rightarrow 0$$

$$r = 0, \dots, (n-1)$$

$$= (-1)^n \frac{\Gamma(p)}{\Gamma(p-n)} \tilde{f}(p-n)$$

(G).

$$\mathcal{M}[xf'(x)] = -p\tilde{f}(p) \rightarrow \text{Exercise}$$

(H).

$$\mathcal{M}[x^2 f''] = (-1)^2 p(p+1)\tilde{f}(p) \rightarrow \text{Exercise}$$

(I). Mellin Transform of Differential Operator:

$$\begin{aligned} \mathcal{M}\left[\left(x \frac{d}{dx}\right)^2 f\right] &= \mathcal{M}\left[\left(x \frac{d}{dx}\right)\left(x \frac{d}{dx}\right) f\right] \\ &= (-1)^2 p^2 f(p) \end{aligned}$$

Here, Used Properties (H) and (G).

(J) $\mathcal{M}\left[\int_0^x f(t) dt\right] = -\frac{1}{p} \tilde{f}(p+1)$

Proof: $F(x) = \int_0^x f(t) dt \Rightarrow F(0) = 0$

$\mathcal{M}(f) = \mathcal{M}[F'] = -(\beta-1) \mathcal{M}[F, p-1]$ (Recall: $\mathcal{M}(f') = -(\beta-1)\tilde{f}(p-1)$)

Replace $(p-1) \leftrightarrow p$

$\Rightarrow \mathcal{M}[F, p] = p \mathcal{M}[F, p-1]$

$\Rightarrow \mathcal{M}(f, p+1) = -p \mathcal{M}(F, p)$ (D)

$\Rightarrow \mathcal{M}[F, p] = -\frac{1}{p} \mathcal{M}(f, p+1)$

LHS $\mathcal{M}\left[\int_0^x f(t) dt\right] = -\frac{1}{p} \tilde{f}(p+1) = \text{RHS}$

ETBC, IIT DELHI

(J).

$$\mathcal{M}\left[\int_0^x f(t) dt\right] = -\frac{1}{p} \tilde{f}(p+1)$$

Proof:

$$F(x) = \int_0^x f(t) dt \Rightarrow F'(x) = f(x) \quad \& \quad F(0) = 0$$

$$\mathcal{M}(f) = \mathcal{M}[F'] = -(p-1)\mathcal{M}[F, p-1] \quad (1)$$

Replace $p-1$ to p :

$$\Rightarrow \mathcal{M}(f, p+1) = -p\mathcal{M}(F, p)$$

$$\Rightarrow \mathcal{M}(F, p) = \frac{-1}{p}\mathcal{M}(f, p+1)$$

$$\text{LHS} = \mathcal{M}\left[\int_0^x f(t)dt\right] = -\frac{1}{p}\tilde{f}(p+1) = \text{RHS}$$

ok, moving on.