Integral Transforms and Their Applications

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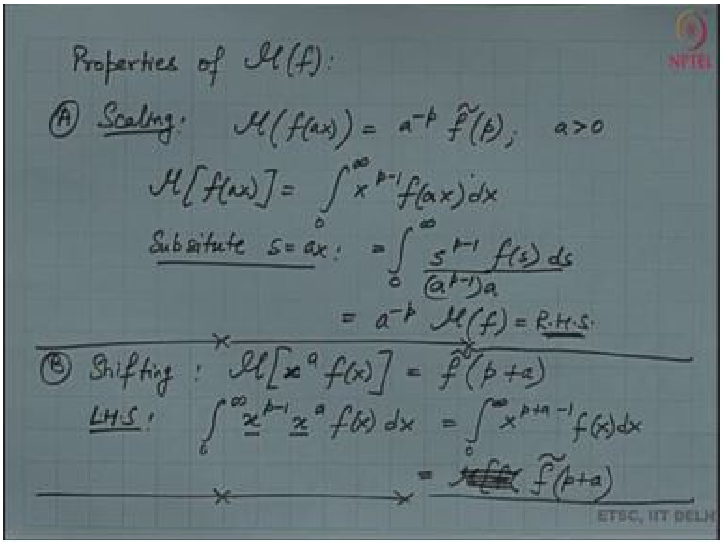
Lecture – 10
Introduction to Mellin Transforms Part - 02

Next I am going to highlight some of the properties of Mellin transform,



(A)Scaling:

$$\mathcal{M}(f(ax)) = a^{-p}\tilde{f}(p); \quad a > 0$$

$$\mathcal{M}[f(ax)] = \int_0^\infty x^{p-1} f(ax)dx$$

Substitute s=ax;

$$= \int_0^\infty \frac{s^{p-1}f(s)ds}{(a^{p-1})\,a}$$

$$= a^{-p}\mathcal{M}(f) = \quad \text{RHS}$$

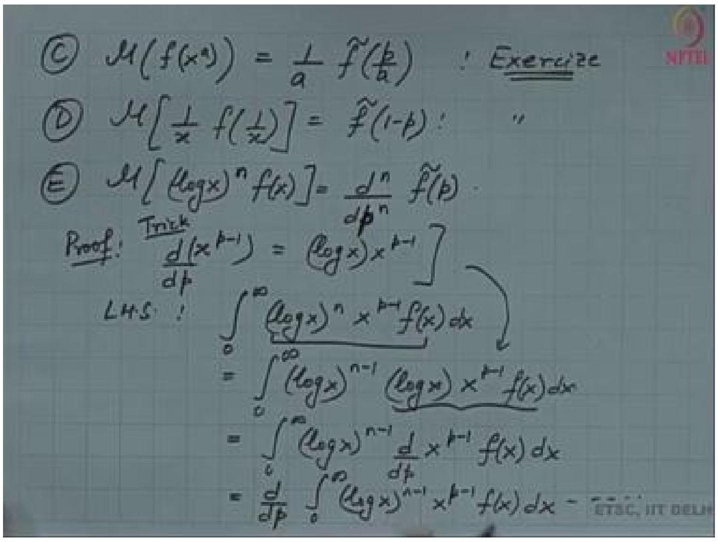
(B).Shifting:

$$\mathcal{M}\left[x^a f(x)\right] = \tilde{f}(p+a)$$

Taking LHS;

$$\int_0^\infty x^{p-1}x^a f(x)dx = \int_0^\infty x^{p+a-1} f(x)dx$$

$$= \tilde{f}(p+a)$$



(C).

$$\mathcal{M}\left(f\left(x^{a}\right)\right)=\frac{1}{a}\tilde{f}\left(\frac{p}{a}\right) \quad : \text{ Exercise}$$

(D).

$$\mathcal{M}\left[\frac{1}{x}f\left(\frac{1}{x}\right)\right]=\tilde{f}(1-p): \quad \text{Exercise}$$

(E).

$$\mathcal{M}\left[(\log x)^{n}f(x)\right]=\frac{d^{n}}{dp^{n}}\tilde{f(p)}$$

Proof:

$$\text{Hint:}\frac{d\left(x^{p-1}\right)}{dp}=(\log x)x^{p-1}$$

LHS:

$$\int_{0}^{\infty}(\log x)^{n}x^{p-1}f(x)dx$$

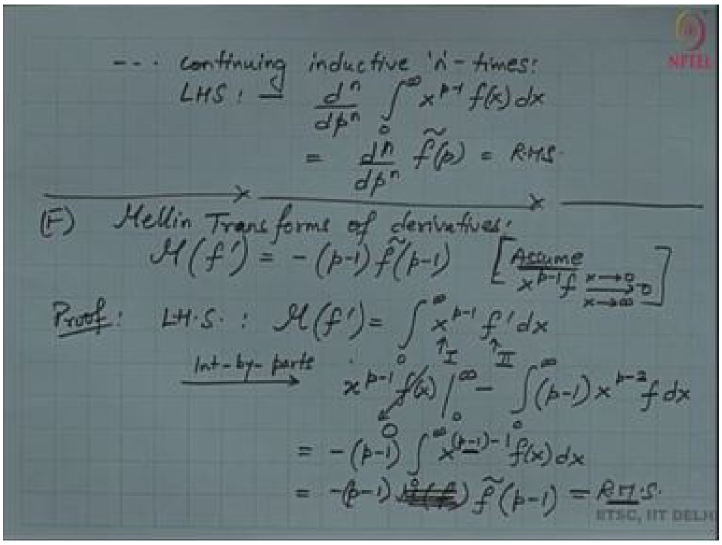
$$=\int_{0}^{\infty}(\log x)^{n-1}(\log x)x^{p-1}f(x)dx$$

$$=\int_{0}^{\infty}(\log x)^{n-1}\frac{d}{dp}x^{p-1}f(x)dx$$

$$=\frac{d}{dp}\int_{0}^{\infty}(\log x)^{n-1}x^{p-1}f(x)dx\text{......continuing inductively n times.}$$

$$LHS:\frac{d^{n}}{dp^{n}}\int_{0}^{\infty}x^{p-1}f(x)dx$$

$$=\frac{d^{n}}{dp^{n}}f(p)=RHS$$



So, this time I am going to highlight Mellin transforms of derivatives. So, Mellin transforms of derivatives are given by

(F). Mellin transforms of derivatives:

$$\mathcal{M}\left(f'\right)=-(p-1)\tilde{f}(p-1)$$

Proof:

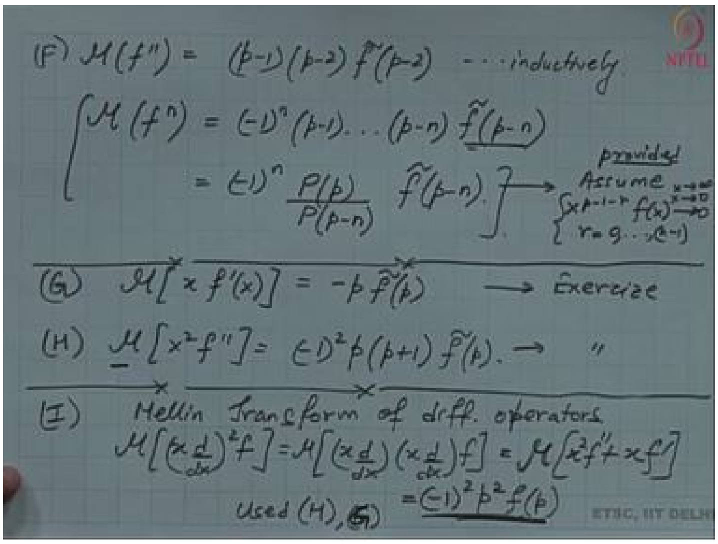
$$\text{LH.S.: } \mathcal{M}\left(f'\right)=\int_{0}^{\infty} x^{p-1} f' dx$$

So, using integration by parts I get

$$x^{p-1} f(x)\Big|_{0}^{\infty}-\int_{0}^{\infty}(p-1) x^{p-2} f dx$$

$$=-(p-1) \int_{0}^{\infty} x^{(p-1)-1} f(x) dx$$

$$=-(p-1) \tilde{f}(p-1)=\text{RHS}$$



$$\mathcal{M}\left(f''\right) = (p-1)(p-2)\tilde{f}(p-2)\ldots \text{inductivly}$$

$$\mathcal{M}\left(f^{n}\right) = (-1)^{n}(p-1)\ldots(p-n)\tilde{f}(p-n)$$

I assume that:

$$x^{p-1-r}f(x) \to 0 \quad \text{as}, x \to 0$$
$$r = 0, ., . \cdot (n-1)$$

$$= (-1)^{n}\frac{\Gamma(p)}{\Gamma(p-n)}\tilde{f}(p-n)$$

(G).

$$\mathcal{M}\left[xf'(x)\right] = -p\tilde{f}(p) \quad \to \text{ Exercise}$$

(H).

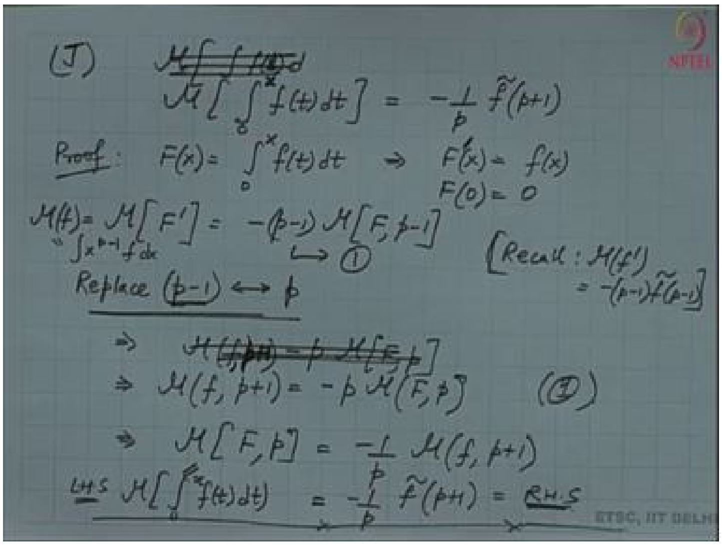
$$\mathcal{M}\left[x^{2}f''\right] = (-1)^{2}p(p+1)\tilde{f}(p) \to \text{Exercise}$$

(I).Mellin Transform of Differential Operator:

$$\mathcal{M}\left[\left(x\frac{d}{dx}\right)^{2}f\right] = \mathcal{M}\left[\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)f\right]$$

$$= (-1)^{2}p^{2}f(p)$$

Here,Used Properties (H) and (G).



(J).

$$\mathcal{M}\left[\int_{0}^{x} f(t)dt\right] = -\frac{1}{p}\tilde{f}(p+1)$$

Proof:

$$F(x) = \int_{0}^{x} f(t)dt \Rightarrow F'(x) = f(x) \quad \& \quad F(0) = 0$$

$$\mathcal{M}(f) = \mathcal{M}\left[F'\right] = -(p-1)\mathcal{M}[F, p-1] \qquad (1)$$

Replace p-1 to p;

$$\Rightarrow \mathcal{M}(f, p+1) = -p\mathcal{M}(F, p)$$

$$\Rightarrow \mathcal{M}(F, p) = \frac{-1}{p}\mathcal{M}(f, p+1)$$

$$\text{LHS} = \mathcal{M}\left[\int_0^x f(t)dt\right) = -\frac{1}{p}\tilde{f}(p+1) = \text{RHS}$$

ok, moving on.