Integral Transforms and Their Applications
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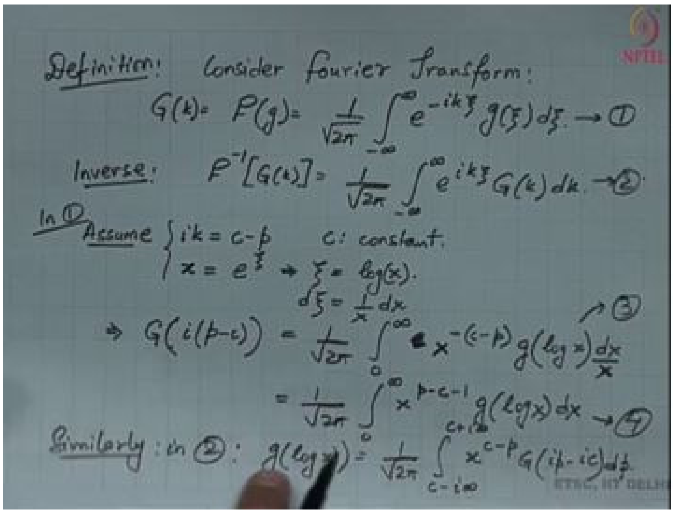
Lecture – 10
Introduction to Mellin Transforms Part - 01

So, today I am going to talk about another new transformation that is the Mellin Transforms. So, Mellin transforms are quite useful in problems related to number theory, in mathematical statistics and especially in evaluating some asymptotic series in the asymptotic limits the series sum.

In fact, in 1876 Remand famously recognized the importance of Mellin transforms in his famous memoirs on prime numbers. So, let us start our lecture and discussion today by deriving the Mellin transforms from our Fourier transform.



So, consider to arrive at the definition of Mellin transform I need to consider Fourier transform. So, consider my Fourier transform

$$G(k) = \mathcal{F}(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ik\xi} g(\xi) d\xi \rightarrow (1)$$

then the inverse the inverse of this transform is defined as

$$\mathcal{F}^{-1}[G(k)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ik\xi} G(k) dk \rightarrow (2)$$

In equation (1):Assume,

$$ik = c - p \quad \text{c: constant}$$

$$x = e^{\xi} \rightarrow \xi = \log(x)$$

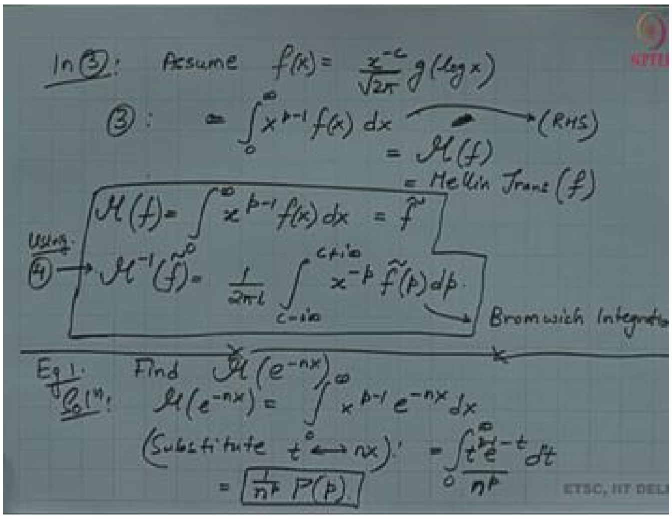
$$d\xi = \frac{1}{x} dx$$

$$\Rightarrow G(i(p-c)) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{-(c-p)} g(\log x) \frac{dx}{x} \qquad (3)$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{p-c-1} g(\log x) dx \rightarrow (4)$$

Similarity in Equ. (2):

$$g(\log x) = \frac{1}{\sqrt{2\pi}} \int_{c-i\infty}^{c+i\infty} x^{c-p} G(ip - ic) dp$$



So, in this expression 3 let us now substitute further. So, in 3 what I have is let me assume another function; assume my function:

$$f(x) = \frac{x^{-c}}{\sqrt{2\pi}} g(\log x)$$

So, I have just clubbed in I have clubbed in most of these terms into a single function here. So, now the 3 becomes the equation

$$\int_0^\infty x^{p-1} f(x) dx = \mathcal{M}(f)$$

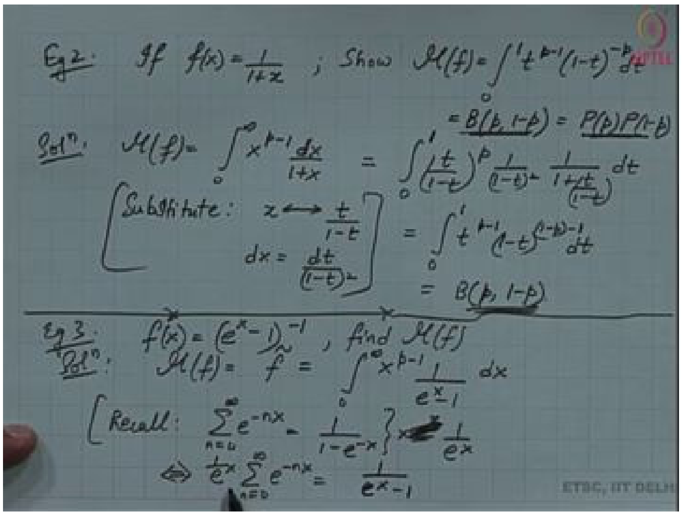
So, I call this as I am now going to define this integral as my Mellin transform of f; so this is my Mellin transform of f.

So, I denote my Mellin transform with this curly M. So, notice in all my in almost all my lectures I am denoting the transform operator or the transform operation by these curly symbols; so the Mellin transform with a $\mathcal{M}$

Using 4:

$$\mathcal{M}(f) = \int_0^\infty x^{p-1} f(x) dx = \tilde{f}$$
$$\mathcal{M}^{-1}(\tilde{f}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-p} \tilde{f}(p) dp$$

So, let us look at another example.



Example 2:

$$\text{If } f(x) = \frac{1}{1+x}; \quad \text{Show } \mathcal{M}(f) = \int_0^\infty t^{p-1}(1-t)^{-p}dt = B(p, 1-p) = \Gamma(p)\Gamma(1-p)$$

Solution:

$$\mathcal{M}(f) = \int_0^\infty x^{p-1}\frac{dx}{1+x}$$

$$\int_0^1 \left(\frac{t}{1-t}\right)^p \frac{1}{(1-t)^2}\frac{1}{1+\frac{t}{(1-t)}}dt$$

$$\text{Substitute: } x \leftrightarrow \tfrac{t}{1-t} \qquad dx = \tfrac{dt}{(1-t)^2}$$

$$= \int_0^1 t^{p-1}(1-t)^{(1-p)-1}dt$$

$$= B(p, 1-p)$$

Example 3:

$$f(x) = (e^x - 1)^{-1}, \text{ find } \mathcal{M}(f)$$

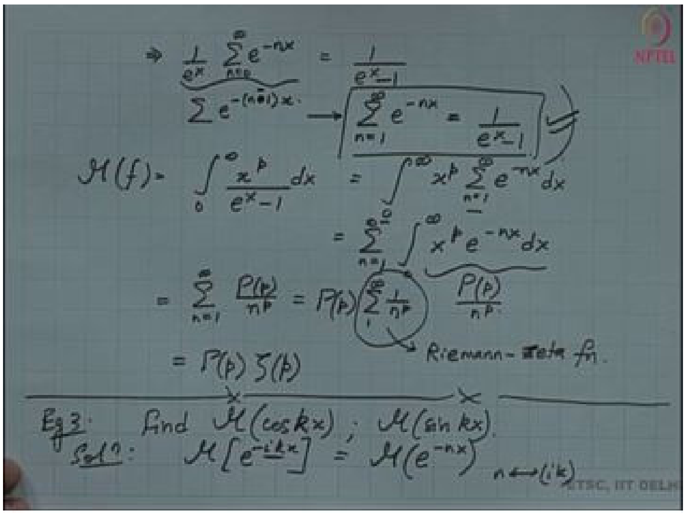
Solution:

$$\mathcal{M}(f) = \tilde{f} = \int_0^\infty x^{p-1}\frac{1}{e^x - 1}dx$$

$$\left\{ \text{Recall : } \sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}} \right\}$$

$$\Rightarrow \frac{1}{e^x}\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{e^x - 1}$$

So, this is the term that the term on the right hand side is what we want and the term on the left hand side is nothing, but notice that this is let me just write this series again.



$$\Rightarrow \frac{1}{e^x}\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{e^x-1}$$

$$\sum_{n=1}^{\infty} e^{-nx} = \frac{1}{e^x-1}$$

$$\mathcal{M}(f) = \int_0^{\infty} \frac{x^p}{e^x-1}dx$$

$$= \int_0^{\infty} x^p \sum_{n=1}^{\infty} e^{-nx}dx$$

$$= \sum_{n=1}^{\infty} \underbrace{x^p e^{-nx} dx}_{\frac{\Gamma(p)}{n^p}}$$

$$= \sum_{n=1}^{\infty} \frac{\Gamma(p)}{n^p} = \Gamma(p)\left(\sum_{1}^{\infty}\frac{1}{n^p}\right)$$

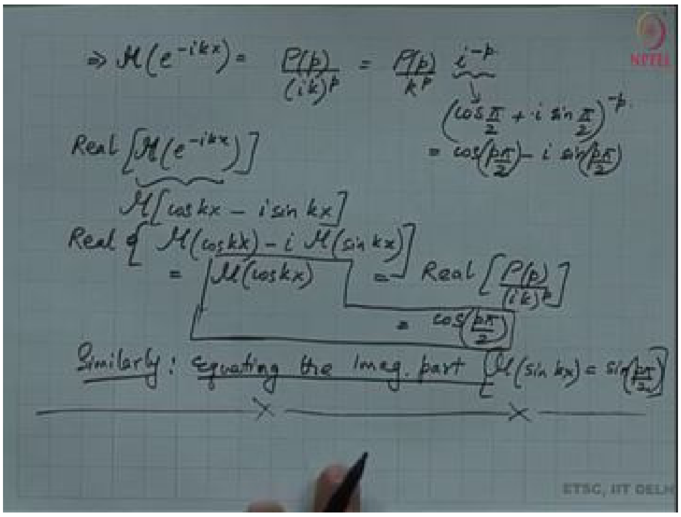
$$= \Gamma(p)\xi(p) \qquad \text{where, } \xi(p) = \text{Riemann zeta function} = \sum_{1}^{\infty}\frac{1}{n^p}$$

Example 3:

$$\text{Find } \mathcal{M}(\cos kx) \quad \text{and} \quad \mathcal{M}(\sin kx))$$

Solution:

$$\mathcal{M}\left[e^{-ikx}\right] = \mathcal{M}\left(e^{-nx}\right) \quad n \leftrightarrow (ik)$$



$$\Rightarrow \mathcal{M}\left(e^{-ikx}\right) = \frac{\Gamma(p)}{(ik)^p} = \frac{\Gamma(p)}{k^p} i^{-p}$$

we know that,

$$i^{-p} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{-p}$$

$$= \cos\left(\frac{p\pi}{2}\right) - i\sin\left(\frac{p\pi}{2}\right)$$

So, if I were to take the real part of this left hand side; I get that this is also equal to

$$\text{Real}\left[\mathcal{M}\left(e^{-ikx}\right)\right]$$

$$= \text{Real}\left[\frac{\Gamma(P)}{(ik)^p}\right]$$

$$\text{Real}\left[\mathcal{M}(\cos kx) - i\mathcal{M}(\sin kx)\right]$$

$$= \mathcal{M}(\cos kx)$$

$$\cos\left(\frac{p\pi}{2}\right)$$

And similarly equating the imaginary part I get that my Mellin transform of sin kx :

$$\mathcal{M}(\sin kx) = \sin\left(\frac{p\pi}{2}\right)$$