Integral Transforms and Their Applications

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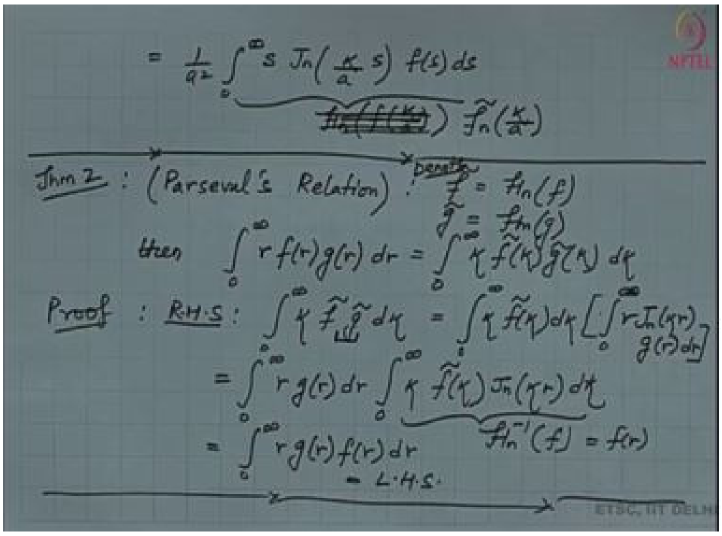
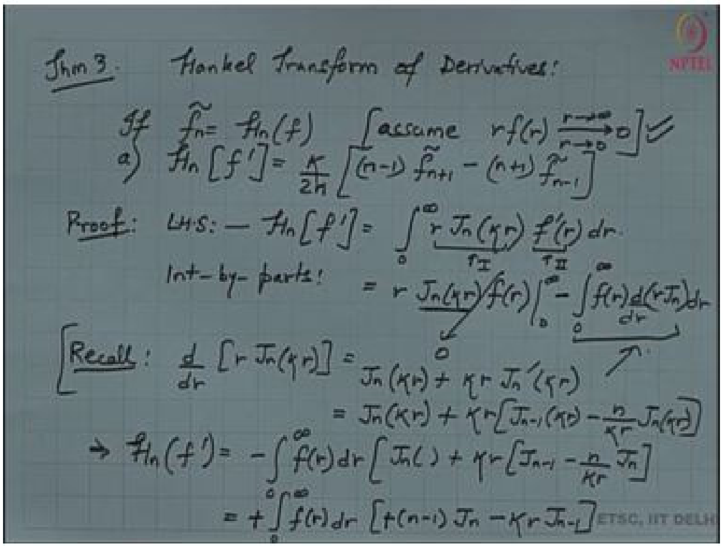
Lecture – 9

Introduction to Hankel Transforms- 02



Let me give you one more result in the form of another theorem.



So, that is about the Hankel Transform of derivatives. So, Hankel Transform Transform of Derivatives ok.

Theorem 3: Hankel tranform of Derivatives:

$$\text{If } \tilde{f}_n = \mathrm{H}_n(f) \text{ [assume , } \quad rf(r)\xrightarrow[r\to 0]{r\to\infty}\Big]\Big\}$$
$$\text{a) } [\mathrm{H}_n'] = \frac{K}{2n}\left[(n-1)\tilde{f}_{n+1} - (n+1)\tilde{f}_{n-1}\right]$$

So, notice that this sort of a recursion relation arises due to the fact that we have a similar recursion relation in the Bessel functions. So, we are going to use the property of the recursion in the Bessel functions to arrive at this recursion of the Hankel Transform.

Proof:

$$\text{LHS:}\quad \mathrm{H}\left[f'\right] = \int_0^\infty rJ_n(\|r)f'(r)dr$$

$$\text{Integration -by-parts!}\qquad = rJ_n(\|r)f(r)|_0^\infty - \int_0^\infty f(r)\tfrac{d}{dr}\left(rJ_n\right)dr$$

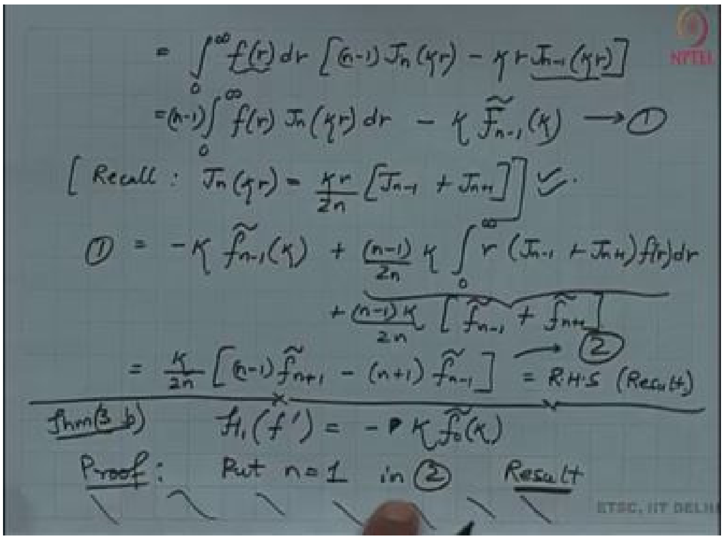
$$\text{Recall: } \frac{d}{dr}[rJ_n(kr)] = J_n(kr) + krJ_n'(kr)$$

$$= J_n(kr) + kr\left[J_{n-1}(kr) - \frac{n}{kr}J_n(kr)\right]$$

$$\Rightarrow H_n\left(f'\right) = -\int_0^\infty f(r)dr\left[J_n() + kr\left[J_{n-1} - \frac{n}{kr}J_n\right]\right]$$

$$= +\int_0^\infty f(r)dr\left[+(n-1)J_n - KrJ_{n-1}\right]$$

So, that is my integral here ok. So, then notice that. So, let me just rewrite and keep processing this result.



$$= \int_0^\infty f(r)dr\left[(n-1)J_n(kr) - KrJ_{n-1}(kr)\right]$$

$$= (n-1)\int_0^\infty f(r)J_n(kr)dr - k\tilde{F}_{n-1}(k) \qquad (1)$$

$$\left[\text{Recall} : J_n\left(kr\right) = \frac{kr}{2n}\left[J_{n-1} + J_{n+1}\right]\right]$$

$$(1) = -\kappa \tilde{f}_{n-1}(k) + \frac{(n-1)}{2n} k \left[\int_0^\infty r(J_{n-1} + J_{n+1}) f(r) dr \right] + \frac{(n-1)k}{2n} \left[\tilde{f}_{n-1} + \tilde{f}_{n+1} \right]$$

$$= \frac{k}{2n} \left[(n-1)\tilde{f}_{n+1} - (n+1)\tilde{f}_{n-1} \right] \qquad (2) \quad : R.H.S$$

So, so that is my that is my RHS and my result ok. So, then there is another quick result. So, let me just say that this is my theorem 3 b which is the quick version of 3

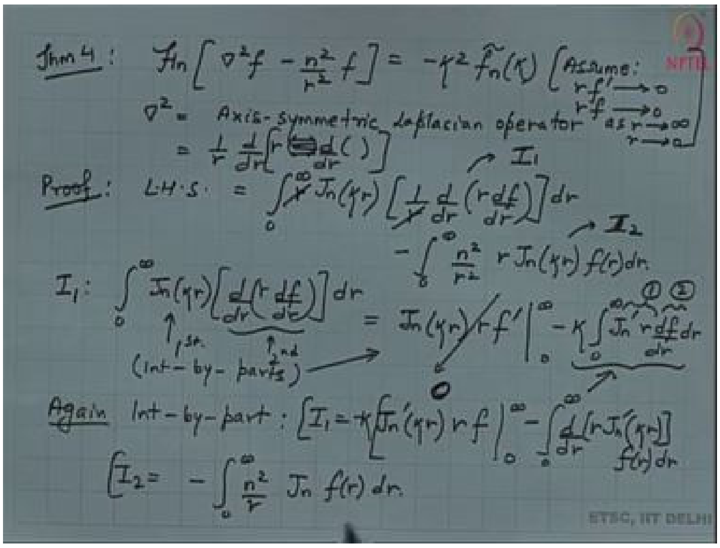
Theorem 3(b):

$$H_1(f') = -\kappa \tilde{f}_0(k)$$

Proof:

Put $n = 1$ in (2) Result

So, I have specially want to denote highlight this particular result because, I will be using this particular version of the theorem, later on in solving application problems moving on, let us now look at one more result again in the form of theorem.



Theorem 4:

$$\mathcal{H}_n \left[\nabla^2 f - \frac{n^2}{r^2} f \right] = -k^2 \tilde{f}_n(k)$$

So, now my Laplacian del square is the axis symmetric. So, it is the axis symmetric version of the Laplacian operator and my Laplacian is given by

$$= \frac{1}{r} \frac{d}{dr} \left[r \frac{d}{dr}() \right]$$

So, let us quickly look at the proof well this last result this particular result theorem 4 will be very useful for us to solve some axis symmetric problems involving Laplacians the axis symmetric Laplacian ok. So, my left hand side is using my definition of Laplace my definition of Hankel Transform is as follows,

Proof:

$$L.H.S = \int_0^\infty r J_n(kr) \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{df}{dr} \right) \right] dr - \int_0^\infty \frac{n^2}{r^2} r J_n(kr) f(r) dr$$

$$I_1(\text{First part of above expression}) : \int_0^\infty J_n(kr) \left[\frac{d}{dr} \left(\frac{df}{dt} \right) \right] dr$$

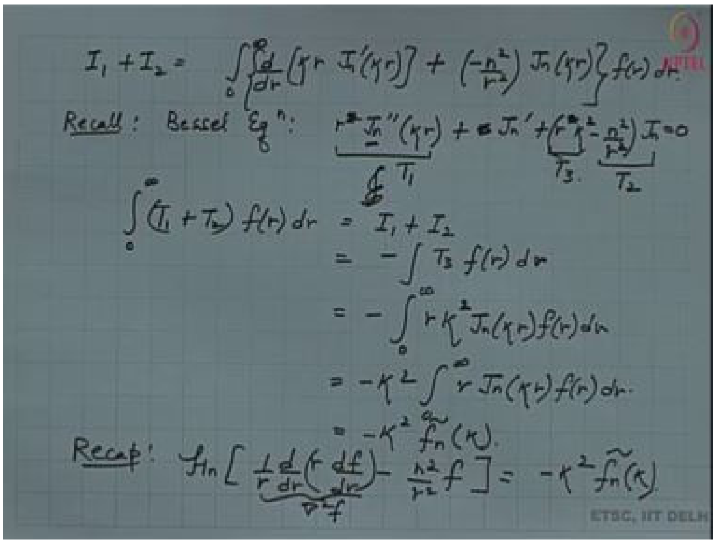
$$= J_n(kr)\, r\, f'|_0^\infty - k \int_0^\infty J_n' r \frac{df}{dr} dr$$

Again Integration by parts:

$$\left[I_1 = -k \left[J_n'(kr)\, r f \right]_0^\infty - \int_0^\infty \frac{d}{dr} \left[r J_n(kr) \right] f(r) dr \right]$$

$$I_2 = - \int_0^\infty \frac{n^2}{r} J_n f(r) dr$$

So, let us look at let us combine I 1 and I 2 and see what happens. So, I get I get after combining I 1 and I 2.



So, I 1 plus I 2, I am going to get the following expression,

$$I_1 + I_2 = \int_0^\infty \left\{ \frac{d}{dr} \left[kr + J_n'(kr) \right] + \left(-\frac{n^2}{r^2} \right) J_n(kr) \right\} f(r) dr$$

So, why I wrote this expression in this form is recall, recall my Bessel function, recall my Bessels equation my Bessels equation ok. So, my Bessels equation looks like following.

$$r J_n'(kr) + J_n' + \left(r k^2 - \frac{n^2}{r^2} \right) J_n = 0$$

Notice that the first term here, if I were to multiply the first time by r. So, this becomes this becomes well let me just show you. So, these two term is nothing, but my integral well let me call this as the first one to be term T 1 and I call this second one to be as my term T 2. So, the integral of the term T 1 plus T 2 times f(r)dr is nothing, but my integral I 1 plus I 2 ok.

$$\int_0^\infty (T_1 + T_2) f(r) dr = I_1 + I_2$$

$$= -\int T_3 f(r) dr$$

$$= -\int_0^\infty r k^2 J_n(kr) f(r) dr$$

$$= -k^2 \int_0^\infty r J_n(kr) f(r) dr$$

$$= -k^2 \tilde{f}_n(k)$$

So, what I have got is the following. So, recap, what I will shown here

$$\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \frac{h^2}{r^2} f\right] = -k^2 \tilde{f}_n(k)$$

So, whenever we have this Laplacian, I can always replace the Laplacian by the two remaining terms in our examples. So, moving on let me just show you some applications.