Integral Transforms and Their Applications
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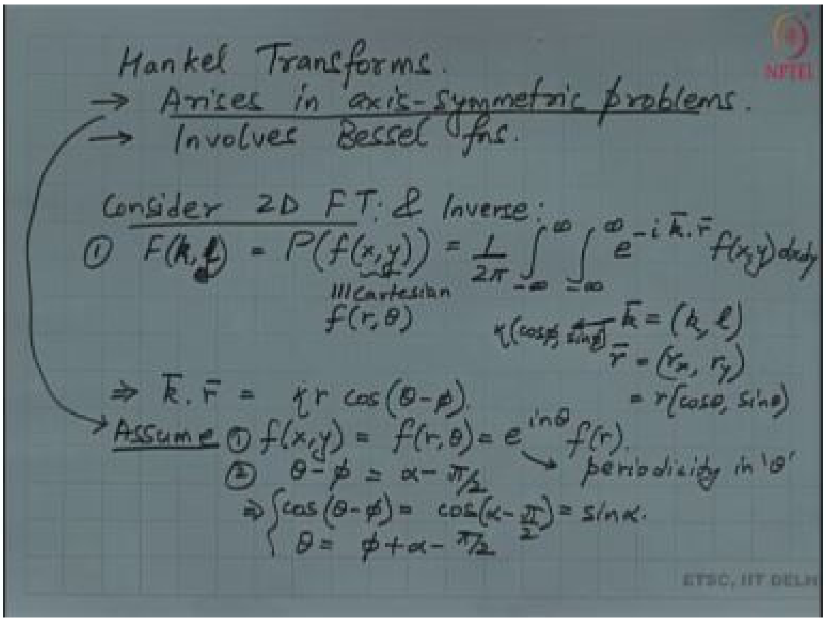
Lecture –09
Introduction to Hankel Transforms Part 1

Good morning everyone. So, just to recap in my last lecture last all my last lectures I have shown you various applications, various properties of Laplace Transform, as well as Fourier Transforms, as well as the applications of joint Fourier and Laplace Transform. Now in today's lecture I am going to focus about is transformation that comes out directly from the Fourier Transform.

So, this particular transform known as the Hankel Transform is a specific case of Fourier Transform and as we will see that the Hankel Transforms are widely applicable for problems involving axis symmetric problems involving axis symmetric equations or equations that do not involve any variation of angle and so, the other issue is that the Hankel Transform involves the so called Bessel functions.



So, let us continue. So, as I mentioned that Hankel Transform will mainly be used to solve axis symmetric problems and we will see as our course of derivation it involves the so called Bessel functions.

So, let us look at the Fourier. So, I am going to revisit my Fourier Transform once more. So, consider the 2DFourier Transform and its inverse. So, my 2DFourier transform in the case of two physical variables,So, moving on my Fourier transform of this function will look like follows.:

$$F(k,l) = P(f(x,y)) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\overline{k}\cdot\overline{r}} f(x,y)dxdy$$

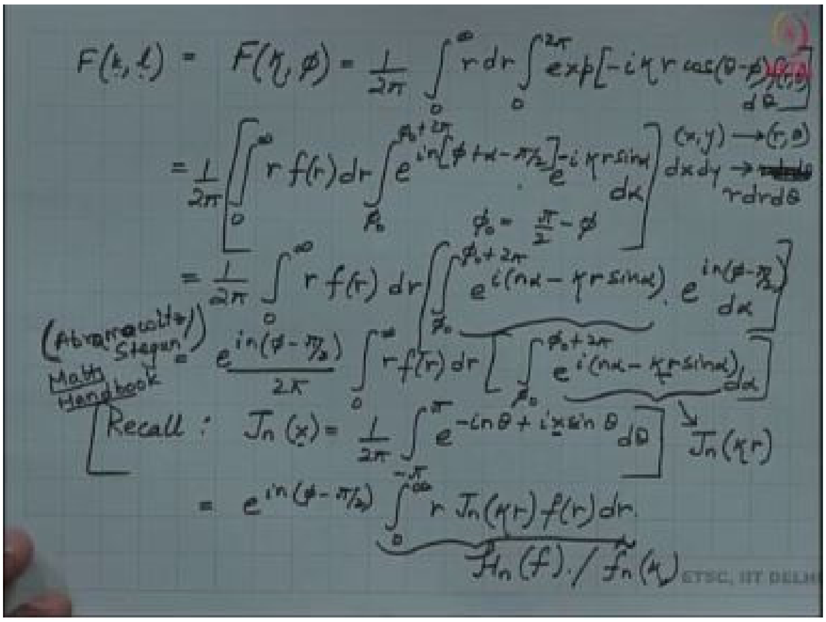
$$\begin{aligned}\overline{k} &= (k,l)\\ \overline{r} &= (r_x, r_y)\\ &= r(\cos\theta, \sin\theta)\end{aligned}$$

$$\overline{k}.\overline{r} = kr\cos(\theta - \phi)$$

Assume:

$$(1)\ f(x,y) = f(r,\theta) = e^{in\theta} f(r)$$
$$(2)\ \theta - \beta = \alpha - \pi/2$$

$$= \begin{cases} \cos(\theta-\phi) = \cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha \\ \theta = \phi + \alpha - \pi/2 \end{cases}$$



So, then this is also equal to using my assumptions that I have made I can assume the periodicity of the function. So, which means I can pull out the function f which is purely a function of r, since we are dealing with the axis symmetric situation and my the theta integral changes as follows:

$$F(k,l) = F(r,\phi) = \frac{1}{2\pi}\int_0^\infty r dr \int_0^{2\pi} \exp[-ikr\cos(\theta - p)(r,\theta)]d\theta$$

$$(x,y) \to (r,\theta)$$

$$= \frac{1}{2\pi}\left[\int_0^\infty rf(r)dr \int_\phi^{\phi+2\pi} e^{in\left[\phi+\alpha-\frac{\pi}{2}\right]} e^{-ikrsin\alpha} d\alpha\right] dxdy \to rdrd\theta.$$

So, we see that now notice that notice this particular integral. So, this involves alpha and this is this particular expression that I have written separately is independent of alpha.

$$= \frac{1}{2\pi}\int_0^\infty rf(r)dr \int_\phi^{\phi+2\pi} e^{i(n_\alpha - kr\sin\alpha)} \cdot e^{in\left(\phi-\frac{\pi}{2}\right)} d\alpha$$

$$= \frac{e^{in(\phi-\pi/2)}}{2\pi}\int_0^\infty rf(r)dr \int_\phi^{\phi+\frac{\pi}{2}} [e^{i(n\alpha - kr\sin\alpha)} d\alpha]$$

So, I am going to frequently refer this book Abramowitz and Stegun, because all the properties the majority of the properties are all listed when we are going to deal with Bessel functions, a lot of properties are all listed in this math handbook. So, recall an expression for the Bessel integral, a Bessel function in the integral form. So, I am going to denote it with this capital letter J n. So, this is a Bessel function of order n. A Bessel function of order n is defined in the integral form its defined as:

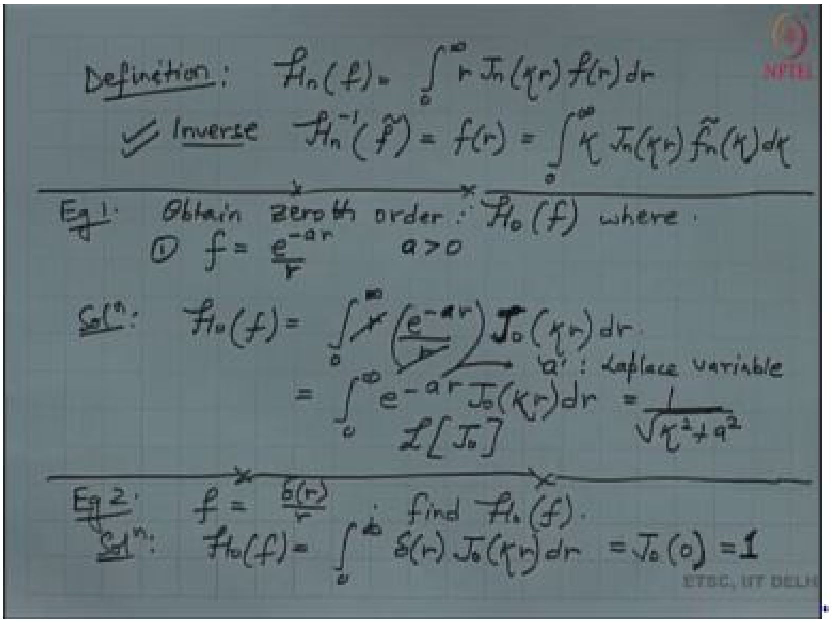
Recall:

$$J_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta + ix\sin\theta} d\theta$$

So, after making suitable change of variables, I can bring my integral to this limit and we can see that this particular integral inside is the Bessel function of order n. It is a Bessel function of order n with argument kappa r, notice that the argument here is x. So, that is the argument x here and hence the argument kappa r that brings in the argument kappa r ok. So, which means my integral looks as follows:

$$= e^{in(\phi - \pi/2)} \int_0^\infty r J_n(kr) f(r) dr$$

Now I am going to define this particular integral as my Hankel Transform, I denote it with this curly H, So, that brings us to the definition of Hankel Transform which will involve this Bessel function. So, again we have to prescribe which order of Hankel Transform are we talking about depending on the order of the Bessel function.



$$H_n(f) = \int_0^\infty r J_n(kr) f(r) dr$$

So, some times I am also going to denote this Hankel Transform by this expression

$$\text{inverse} \quad H_n^{-1}(\hat{f}) = f(r) = \int_0^\infty k J_n(kr) \tilde{f}_n(k) dk$$

So, as we have shown how we get the Hankel Transform from Fourier Transform, it can also be shown that we get this inverse Hankel transform from inverse Fourier transform ok. So, moving on let us look at some examples, on how to use this Hankel transforms. So, the question says

Example 1:

Obtain zeroth order: $H_0(f)$ where
(1) $f = \frac{e^{-ar}}{r} \quad a > 0$

Solution:

$$H_0(f) = \int_0^\infty r \left(\frac{e^{-ar}}{r}\right) J_0\left(kr\right) dr$$

Notice that this particular integral is nothing but a Laplace transform of the Bessel function right where this is nothing but the Laplace transform of the Bessel function of J 0 where my Laplace variable is a. So, my a is my Laplace variable. So, look at the definition of Laplace transform and students are going to immediately recognize that this is the Laplace transform of J 0 and this particular transform, the Laplace transforms problem was done in one of my previous lectures and I get the solution to be again directly from my previous lecture as 1

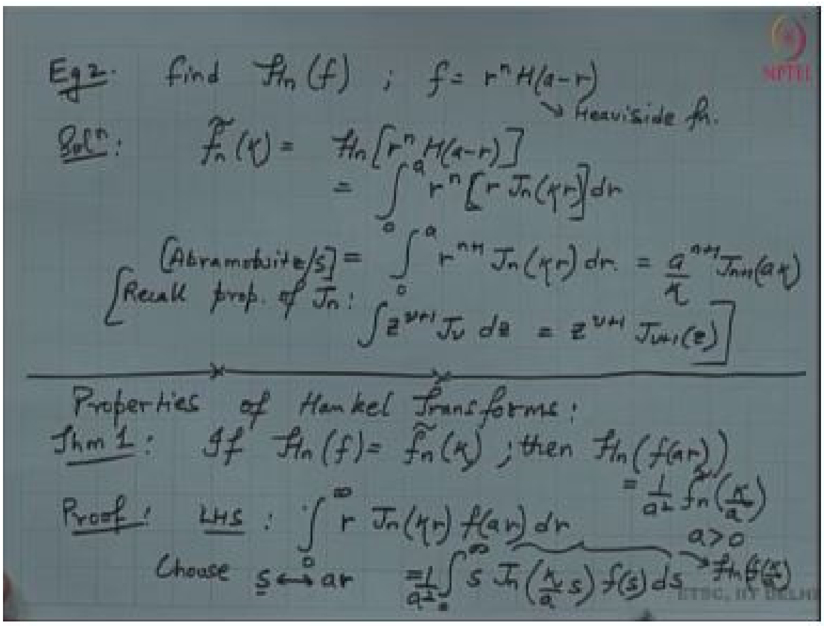
$$= \int_0^\infty e^{-ar} \cdot J_0(kr) dr = \frac{1}{\sqrt{k^2 + a^2}}$$

Example 2:

$$f = \frac{\delta(r)}{r}, \text{ find } H_0(f)$$

Solution:

$$H_0(f) = \int_0^\infty \delta(r) J_0(kr) dr = J_0(0) = 1$$



So, then moving on, let me give you another show you another example, this time find the n ,

$$\text{find } H_n(f); f = r^n H(a - r)$$

Solution:

$$\tilde{f}_n(t) = H_n\left[r^n H(a - r)\right]$$
$$= \int_0^a r^n \left[r J_n(kr)\right] dr$$

$$= \int_0^a r^{n+1} J_n\left(kr\right) dr = \frac{a^{n+1}}{k} J_{n+1}(ak)$$

So, then let me let me recall, let me recall another property of Bessel function property of J n. So, again this is coming from Abramowitz and Stegun the handbook, if people Abramowitz and Stegun. So, people who want to recall these properties, all these properties are listed in this big handbook. So, I am going to recall this property that integral of the Bessel function:
Recall:

$$J_n : \int z^{\nu+1} J_\nu dz = z^{\nu+1} J_{v+1}(z)$$

So, I am now going to look at some properties, some properties of Hankel transform with which I am going to work, through which I am going to work on some harder problems involving Hankel transforms. So, the first property says,

Properties of Hankel transform:

Theorem 1:

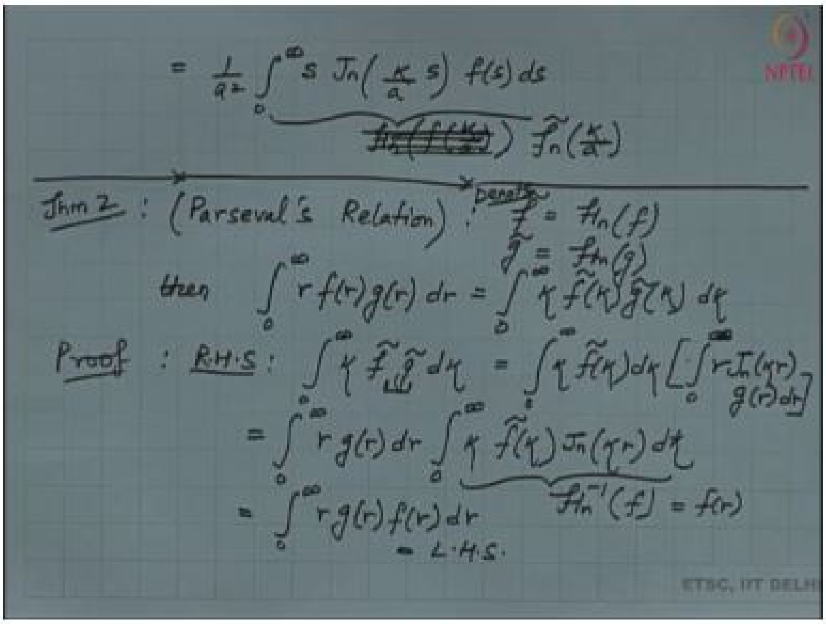
If $H_n(f) = f_n(k)$,then $H_n\left(f\left(ar\right)\right)$= $\frac{1}{a^2}\,\mathrm{f_n}\left(\frac{k}{a}\right)$

Proof:

$$\int_0^\infty rJ_n(kr)f(ar)dr$$

Choose $S \leftrightarrow ar$ Notice that this is nothing but this particular integral is nothing but the Hankel transform of the function at k/a.

$$= \frac{1}{a^2}\int_0^\infty sJ_n\left(\frac{k}{a}s\right) f(s)ds$$



So, moving on I have another property, let me just continue denoting these properties in terms of theorems. So, this particular theorem is also known as the Parseval's relation. So, we have seen a Parseval's relation in Fourier transform. So, this is a specific case of that Parseval's relation related to axis symmetric problems. It again says that if I am given:

Theorem 2:

Parseval's Relation:

If $f = H_n(f)$ and $g = H_n(g)$ then $\int_0^\infty rf(r)g(r)dr = \int_0^\infty k\tilde{f}(k)\tilde{g}(k)dk$

So, I am going to, let us now keep one of these Hankel transforms aside and let us expand the second one using the definition of Hankel Ttransform.

Proof:

$$RHS = \int_0^\infty k\tilde{f}(k)dk[\int_0^\infty rJ_n(kr)g(r)dr]$$
$$= \int_0^\infty rg(r)dr \int_0^\infty kf(\tilde{k})J_n(kr)dk$$
$$= \int_0^\infty rg(r)f(r)dr = LHS$$

So, moving on.