Integral Transforms and Their Applications
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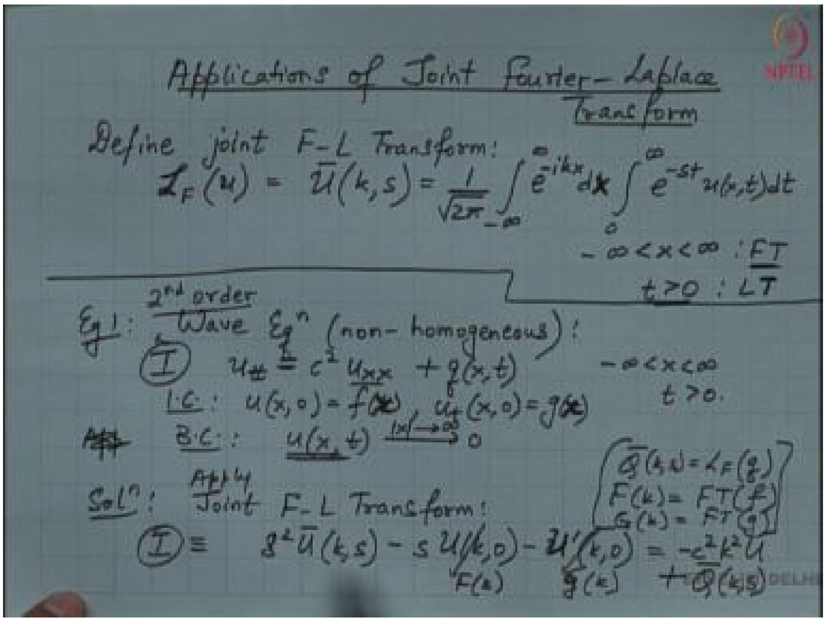
Lecture – 08
Applications of Fourier-Laplace Transforms Part 2



So, now I am going to talk about, I am going to talk about now applications of joint Fourier Laplace Transforms. So, when we have both time and space and both can be transformed, let us look at the scenario where the solution can be found using both the Fourier and the Laplace Transform. So, let us define; let us define my joint Fourier Laplace transform:

$$L_F(u) = \overline{u}(k,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} dx \int_0^{\infty} e^{-st} u(x,t) dt$$
$$-\infty < x < \infty : FT$$
$$t > 0 : LT$$

So, here typically we need that x varies from minus infinity to infinity. So, that is the, whichever variable varies for all values that is the candidate for Fourier transform. And whichever values take only the positive values that is the candidate for Laplace transform. So, this must be our decision decide deciding factor which one to choose for the Fourier variable and which one to choose for the Laplace variable. So, let us look at an example. So, consider the wave equation : Example 1:

Wave Equation (Non-homogeneous):

$$u_t = c^2 u_{xx} + q(x,t) \qquad -\infty < x < \infty$$
$$IC : u(x,0) = f(x), u_t(x,0) = g(x) \qquad t > 0$$

So, these are my initial condition and boundary condition:

$$BC : u(x,t) \overset{|x|\to\infty}{\longrightarrow} 0$$

So, that is what is required for the Fourier Transform of this variable to exist. So, let us take this as the boundary condition. So, then finding the solution will involve taking, the joint. So, apply the joint Fourier Laplace transform. So, we know where to apply the Fourier transform namely the variable x, because of this choice, and where to apply the Laplace Transform because of this choice.

Solution:

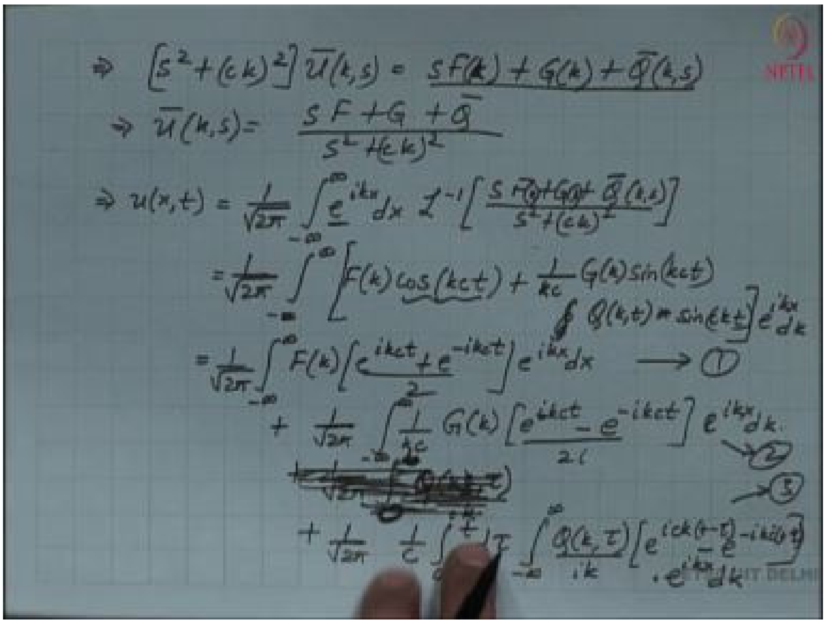
Let:

$$\begin{array}{l} Q(k,s) = L_F(q) \\ F(k) = FT(f) \\ G(k) = FT(q) \end{array}$$

Apply Joint F-L transform,

$$(I) \equiv \quad S^2\overline{u}(k,s) - su(k,0) - u'(k,0) = -c^2k^2u + \overline{Q}(k,s)$$

So, $\overline{Q}(k,s)$ is the joint Fourier Laplace Transform of non homogeneous term Q here. So, let us now solve this equation and see what is the answer.



So I get the following expression:

$$\begin{array}{l} \Rightarrow [s^2 + (ck)^2]\,\overline{u}(k,s) = sF(k) + G(k) + \overline{Q}(k,s) \\ \Rightarrow \overline{u}(k,s) = \frac{sF+G+\overline{Q}}{s^2+(ck)^2} \end{array}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} dx L^{-1} \left[\frac{sF(k) + G(k) + \overline{Q}(k,s)}{s^2 + (ck)^2} \right]$$

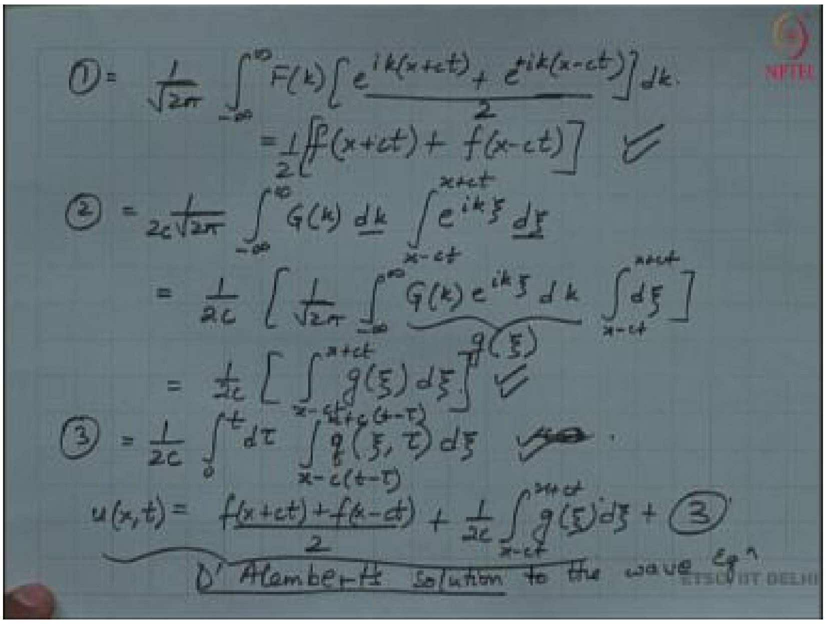
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[F(k)\cos(kct) + \frac{1}{kc} G(k)\sin(kct) Q(k,t) * sin(ckt) \right] e^{ikx} dk$$

So, I can always find the Laplace inverse.So, just taking, the Laplace continuing to take the Laplace Transform and then. So, then the third term, when we take the inverse and we notice that this is, So, I am going to use my Euler's formula for cos and sin and I get that the first expression :

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} F(k)[\frac{e^{ikct} + e^{-ikct}}{2}]e^{ikx}dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{kc} G(k) \left[\frac{e^{ikct} - e^{-ikct}}{2i}\right] e^{ikx}dx$$

$$+\frac{1}{\sqrt{2\pi}}\frac{1}{c} \int_0^t d\tau \int_{-\infty}^{\infty} \frac{Q(k,\tau)}{ik} \left[e^{ick(t-\tau)} - e^{-ikc(t-\tau)}\right] e^{ikx}dk$$

So, those are the three integral, this one 1, this one 2, and this one 3, corresponding to these three terms ok. So, then let us start evaluating the integrals, I get the first one; the first one is quite easy.



$$(1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \left[e^{ik(x+ct)} + e^{+ik(x-ct)}\right] dk = \frac{1}{2} \left[f(x+ct) + f(x-ct)\right]$$

So, that is the first integral. The second integral gives :

$$\begin{aligned} (2) &= \frac{1}{2c\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k)dk \int_{x-ct}^{x+ct} e^{ik\xi} d\xi \\ &= \frac{1}{2c} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(k)e^{ik\xi}dk \int_{x-ct}^{x+ct} d\xi\right] \\ &= \frac{1}{2c} \left[\int_{x-ct}^{x+ct} g(\xi)d\xi\right]^2 \end{aligned}$$

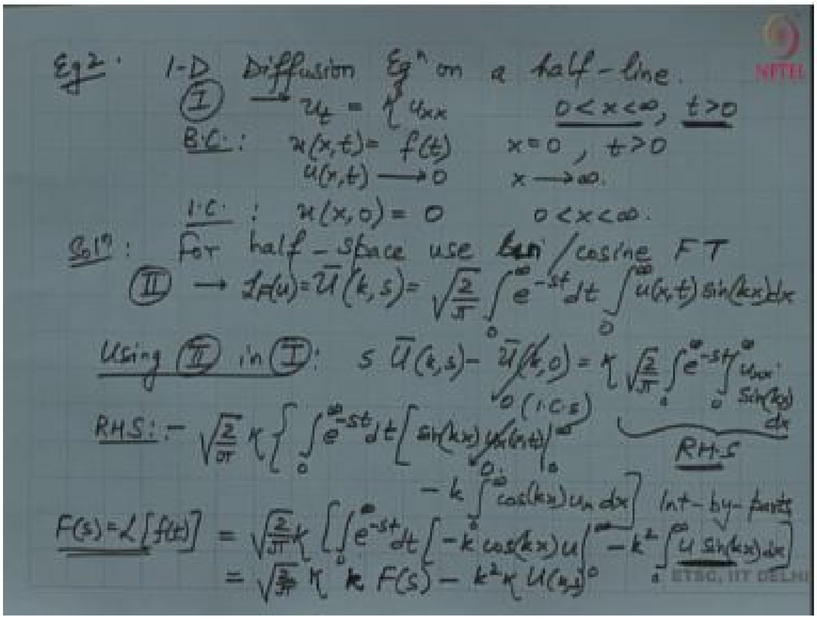
So, I get that this is also equal to the Fourier transform of well. This is the Fourier transform of inverse:

$$(3) \ = \frac{1}{2c} \int_0^t d\tau \int_{x-c(t-\tau)}^{x^{x+c(t-\tau)}} g(\xi,\tau)d\xi$$

what I see is that this also brings in the integral of Q which was the inverse transform of capital Q of this. So, the integral looks like the following. So, I am combining all my integrals into a neat single integral here.So, that is what were the integral cannot be evaluated further, and I leave this third expression as it is. So, combining this result, and this result, and this result I arrive at my solution :

$$u(x,t) = \frac{f(x+ct)+f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi + (3)$$

So, this is the solution to the wave equation, and students can easily find that this is the well known, D'Alemberts solution to the wave equation. So, specifically if we are we are given certain initial conditions, let us say the Dirichlet condition as well as the initial conditions for u, as well as u(t), as well as the case when the solution is the equation is non homogenous, each of these terms will either appear or disappear. So, this is the famous D Alemberts solution, to the wave equation.



Moving on, let us now look at the case were. We have again we are revisiting the case of 1 D Diffusion Equation, on a half-half line. So, we see that the equation to be solved is :

Example 2:

$$(1) \rightarrow u_t = \{\ ku_{xx} \quad 0 < x < \infty, t > 0$$

So, the natural choice to use is either the Fourier cosine or the Fourier sine transform. So, we will not be using the full Fourier transform, but just the Fourier cosine or sine transform. Boundary conditions and Initial conditions are:

$$\begin{array}{c} \text{BC:} \quad u(x,t) = f(t) \quad x = 0, t > 0 \\ u(x,t) \rightarrow 0 \quad x \rightarrow \infty \\ \text{IC:} \quad x(x,0) = 0 \quad 0 < x < \infty \end{array}$$

So, as I said for half space the general rule of thumb is to use, either Fourier cosine or Fourier sine series. So, let me say Fourier sine or Fourier cosine series : Solution:

$$(2) \rightarrow L_F(u) = \overline{u}(k,s) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-st} dt \int_0^\infty u(x,t) \sin(kx) dx$$

So, use using II in I, what I get is the following expression. So, I get I apply the Laplace transform:

Applying 2 in 1:

$$s\overline{u}(k,s) - \overline{u}(k,0) = k\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-st} \int_0^\infty u_{xx} sin(kx) dx.$$

So, my right hand side is the following:

$$\sqrt{\frac{2}{\pi}} k \left[\int_0^\infty e^{-st} dt [\sin(kx) u_x(xt)|_0^\infty] \ -k \int_0^\infty \cos(kx) u_x dx \right].$$

$$F(s) = L\{f(t)\} = \sqrt{\frac{2}{\pi}} K \left[\int e^{-st} dt \left[-k\cos(kx) u \int_0^\infty \ -k^2 \int_0^\infty u \sin(ux) dx \right] \right].$$

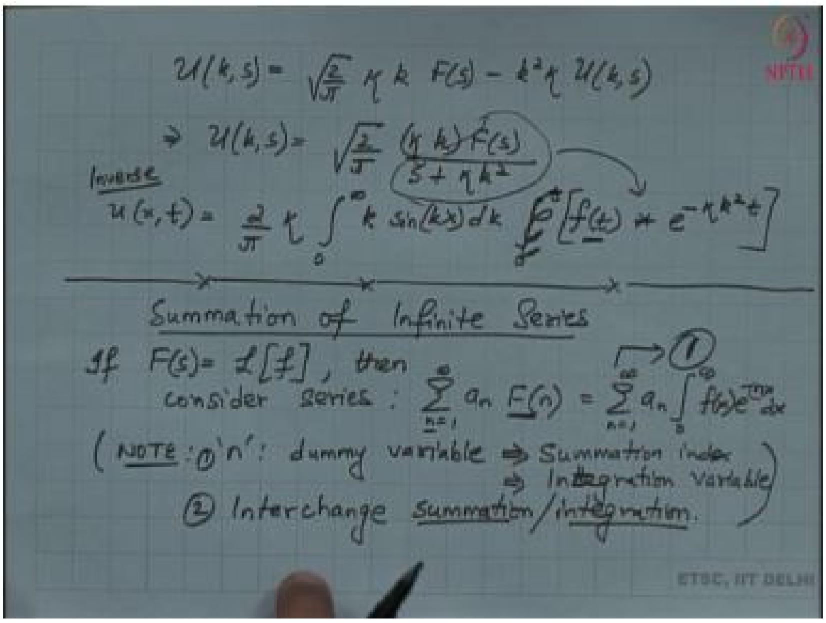
So, again if I apply integration by parts:

$$= \sqrt{\frac{2}{\pi}} KkF(s) - k^2 kU(k,s)$$

So, that is what this integral here, towards the end is given by this u. So, then. So, what I have got is the following.:

$$u(k,s) = \sqrt{\frac{2}{\pi}} kkF(s) - k^2 ku(k,s)$$

$$u(k,s) = \sqrt{\frac{2}{\pi}} \left(\frac{kk)F(s)}{s + kk^2} \right.$$



So, that is where I will end the solution to this problem, because we cannot solve further, without having prior knowledge about this boundary condition f(t) s.

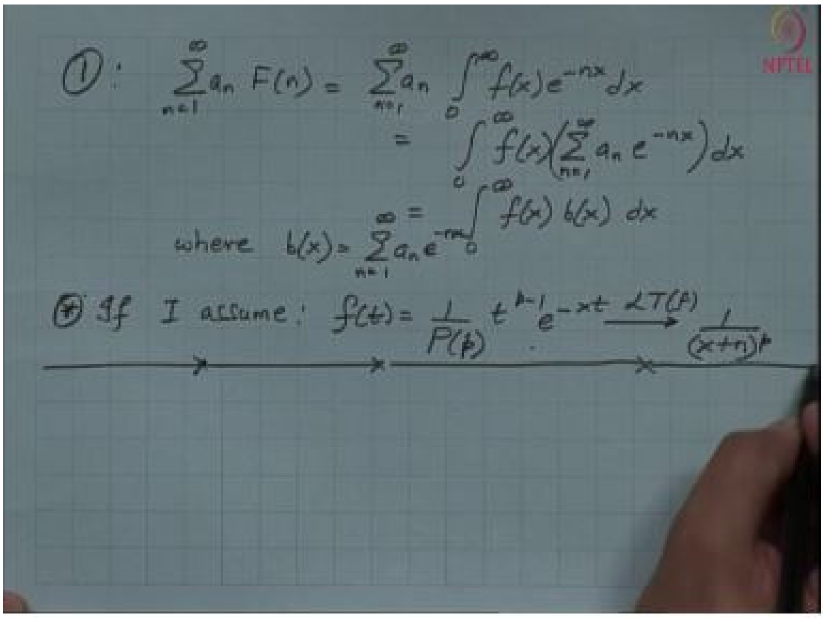
$$u(x,t) = \frac{2}{\pi} k \int_0^{\infty} k \sin(kx) dk \left[f(t) * e^{-kk^2 t} \right]$$

So, now let us look at, let us look at now, another topic that is related to summation of infinite series. So, let us see what mean by that. So, suppose in many situations, we have to sum up infinite series, now the summation becomes very easy if we are to use the Laplace transform. And take the transform of the individual elements of the series. And then simplify that series. So, let us see what I mean by that. So, let us say if I have, F of s it is given to be the Laplace transform of this small f.

So, then consider this series:

If $F(s) = L[f]$, then consider series: $\sum_{n=1}^{\infty} a_n F(n) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} f(x) e^{-nx} dx$

So, we know that the regular definition of Laplace transform s could be complex, but here I am treating n to be, a dummy variable which is acting like s in our Laplace transform. So, then also note that, I can inter change summation and integration. So, what do I mean by that? Due to the fact that the Laplace transform of the function is uniformly convergent I can change my summation and integration. And that will really simplify our calculation :



So, let us denote this by our expression 1 :

$$(1) = \sum_{n=1}^{\infty} a_n F(n) = \sum_{n=1}^{\infty} a_n \int_0^{\infty} f(x) e^{-nx} dx$$

$$= \int_0^{\infty} f(x) \left(\sum_{n=1}^{\infty} a_n e^{-nx} \right) dx$$

$$= \int_0^\infty f(x)d(x)dx$$

$$\text{where } b(x) = \sum_{n=1}^{\infty} a_n e^{-nx}$$

So, the Laplace transform of this quantity is:

If I assume:

$$f(t) = \frac{1}{P(p)} t^{p-1} e^{-xt} \xrightarrow{LT(t)} \frac{1}{(x+n)^p}$$

So, let us using this particular situation, let us try to sum up some series.