**Integral Transforms and Their Applications**
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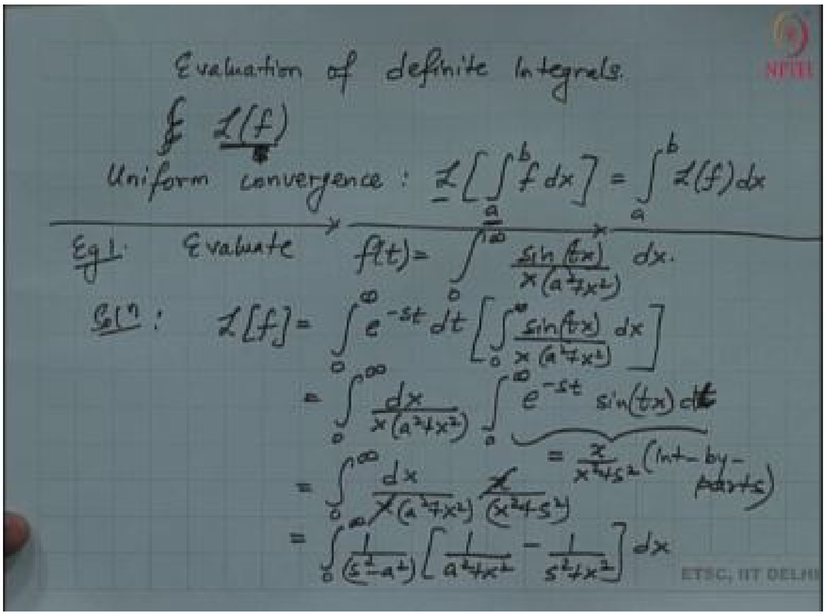
**Lecture – 07**
**Applications of Laplace Transforms (Continued) Part 3**




So, evaluation of definite integrals, so I know that the Laplace transform of the Laplace transform of f, exists for all functions which are of exponential order and the Laplace transform is uniform the transformer is uniformly continuous, due to 1 of the previous results that I have stated. So, due to uniform convergence, due to the fact that, we have uniform convergence of the integral in the Laplace transform, I can always replace or I can always exchange my integral within the Laplace transform and replace or interchange the integral of the Laplace transform with the integral of this particular definite integral.

Mainly what I am saying is that, I can always take the transform of f first and then evaluate

$$L\left[\int_a^b f dx\right] = \int_a^b L(f) dx$$

So, let us see what I am saying with some examples,

Example 1:

Evaluate:

$$f(t) = \int_0^\infty \frac{\sin(tx)}{x\left(a^2+x^2\right)} dx$$

So, to find this integral let me just look at the Laplace transform of this function. So, the Laplace transform of f is given by the following integral,So, I am going to replace, I am going to interchange the 2 integrals to get the following expression::

$$L[f] = \int_0^\infty e^{-st} dt \left[\int_0^\infty \frac{\sin(tx)}{x\left(a^2+x^2\right)} dx\right]$$

I am interchanging these 2 integrals, the 1 due to dt and the other 1 the integral over x as follows. So, I get the following expression:

$$= \int_0^\infty \frac{dx}{x\left(a^2+x^2\right)} \int_0^\infty e^{-st}\sin(tx)dt$$

So, then the second integral can be found, that this is equal to the following value and we have used integration by parts. So, use integration by parts, to arrive at the result of this interior integral. So, what I get is that, this integral becomes:

$$\int_0^\infty \frac{dx}{x\left(a^2+x^2\right)} \frac{x}{\left(x^2+s^2\right)}$$

So, I can cancel some terms out and then I used the method of partial fractions to arrive at the following expression:

$$\int_0^\infty \frac{1}{\left(s^2-a^2\right)}\left[\frac{1}{a^2+x^2} - \frac{1}{s^2+x^2}\right] dx$$

We see so this quantity sitting outside this bracket is a constant with respect to x. So, all I have to do is to integrate these 2 expressions with respect to x and I see that I get the following result :

$$\frac{1}{s^2-a^2}\left[\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) - \frac{1}{s}\tan^{-1}\left(\frac{x}{s}\right)\right]_0^\infty$$

$$= \frac{1}{s^2-a^2}\frac{\pi}{2}\left[\frac{1}{a}-\frac{1}{s}\right]$$

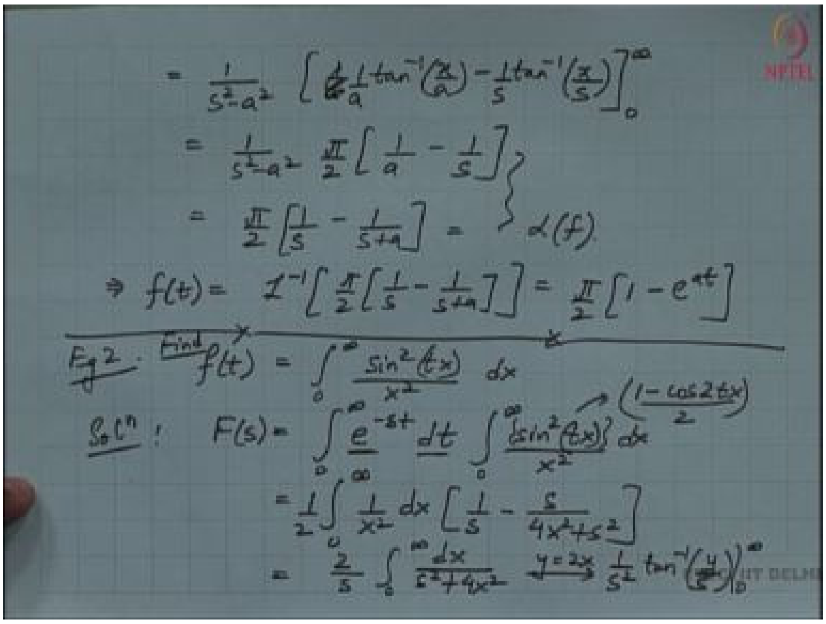
$$= \frac{\pi}{2}\left[\frac{1}{s}-\frac{1}{s+a}\right] = L(f)$$

after all the simplification, I get the following expression:

$$f(t) = L^{-1}\left[\frac{\pi}{2}\left[\frac{1}{s}-\frac{1}{s+a}\right]\right]$$

Now, so which means to this is right now, the Laplace transform of f. So, to find f of t, I need to take the inverse transform of this expression, to arrive at the result as follows:

$$= \frac{\pi}{2}\left[1-e^{at}\right]$$



So, moving on, let us look at another definite integral, question:

Example 2:

Find:

$$f(t) = \int_0^\infty \frac{\sin^2(tx)}{x^2} dx$$

So, notice that if I were to apply the Laplace transform, I get the following expression,again interchanging the 2 integrals, I get that, well even before that I have the following:

$$F(s) = \int_0^\infty e^{-st} dt \int_0^\infty \frac{\sin^2(tx)}{x^2} dx^2$$

I can write this as:

$$\sin^2(tx) = \frac{1 - cos^{2tx}}{2}$$

So, I am going to replace my sin with this expression.So, then what I get is, that this becomes well, what I get is that :

$$\frac{1}{2} \int_0^\infty \frac{1}{x^2} dx \left[ \frac{1}{s} - \frac{s}{4x^2 + s^2} \right]$$

So, all I have done is to evaluate this integral with respect to t, to arrive at this Laplace transform. So, after all the simplification I get the following expression:

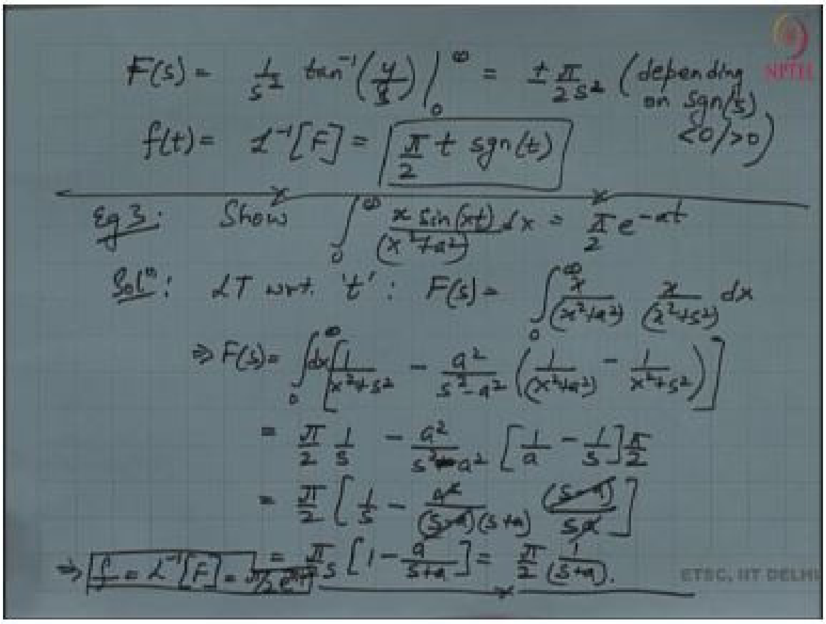
$$\frac{2}{s} \int_0^\infty \frac{dx}{s^2 + 4x^2}$$

And then after suitable transformation, let us choose $y = 2x$, I arrive at the following answer:

$$\frac{1}{s^2} \tan^{-1} \left( \frac{y}{s} \right) \Big|_0^\infty$$

So, then when we evaluate the answer, I get :

$$F(s) = \frac{1}{s^2} \tan^{-1} \left( \frac{y}{s} \right) \Big|_0^\infty = \pm \frac{\pi}{2s^2}$$




depending on the sin of S. Whether it is negative or it is positive. So, if it is positive, we will keep the positive sign and negative we will keep the negative sign. So, which means my inverse transform f of t will be the L inverse or the Laplace inverse :

$$f(t) = L^{-1}[F] = \frac{\pi}{2} t \sin(t)$$

So, then let us look at another quick example. So, the question says :

Example 3:

Show:

$$\int_0^\infty \frac{x \sin(xt)}{(x^2 + a^2)} dx = \frac{\pi}{2} e^{-\alpha t}$$

So, what we have is, again quickly we take the Laplace transform with respect to t and I get that :

$$F(s) = \int_0^\infty dx \frac{(1)}{(x^2 + s^2} - \frac{a^2}{s^2 - a^2} \left( \frac{1}{x^2 + a^2)} - \frac{1}{x^2 + s^2} \right) \Bigg]$$

So, students should check that indeed we are getting this expression using the partial fraction. So, this all simplifies into these 2 factors here ok. So, why I did this because, I can separately evaluate this integral with respect to X and also these 2 integrals with respect to X because, they are going to give me the tan inverse function. So, after doing that I am going to directly write the answer, I get that:

$$= \frac{\pi}{2} \frac{1}{s} - \frac{a^2}{s^2 - a^2} \left[ \frac{1}{a} - \frac{1}{s} \right] \frac{\pi}{2}$$

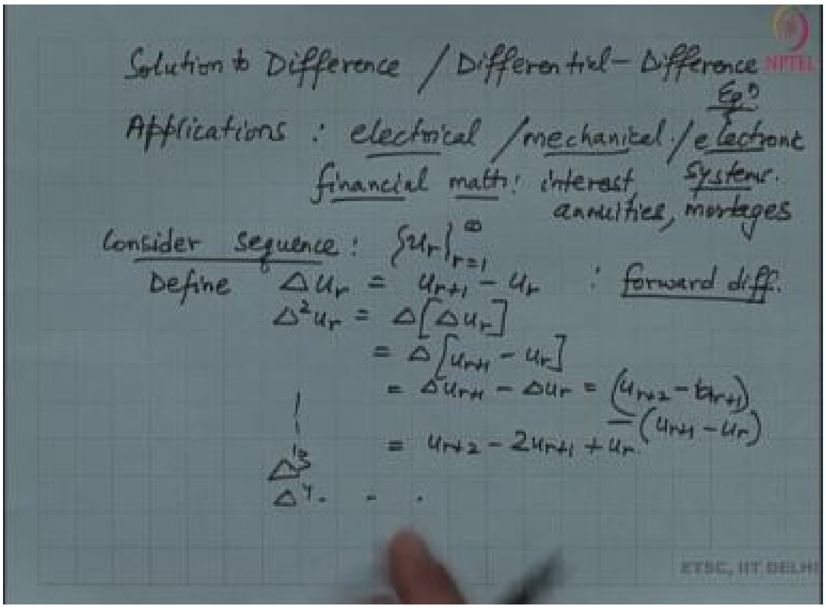
So, I can cancel few factors and I get that answer is :

$$= \frac{\pi}{2} \left[ \frac{1}{s} - \frac{a^2}{(s-a)(s+a)} \frac{(s-a)}{sa} \right]$$

$$= \frac{\pi}{s} \left[ 1 - \frac{a}{s+a} \right] = \frac{\pi}{2} \left( \frac{1}{s+a} \right)$$

So

$$f(t) = L^{-1}[F] = \frac{\pi}{2}e^{-at}$$




So, then I am going to start another sub topic, that is the solution to the difference equation now, difference or differential-difference equation ok. So, before I move ahead I want to highlight. So, in most of the problems involving numerical analysis, we quite often deal with finite difference finite volume finite element. So, essentially we are trying to discretize our continuous equations or a continuous partial or ODEs or Ordinary Differential Equations, using some discretization schemes. So, in those schemes we often encounter these differences. Now, can we solve this differences equation using Laplace transform? Let us have a look at it.

So, we will see that the application of these difference or differential equations, are white in many fold in areas of electrical engineering or you have in areas of mechanical engineering or in electronic systems or even in financial maths, specially when we have to calculate the interest or annuities or mortgages. So, in most of these scenarios, we have to discretize our equations into these discrete forms that I am going to described next.

So, before I go ahead, let us consider some sequence. So, consider some sequence u of r from r from 1 to infinity as follows :

$$\{u_r\}_{r=1}^{\infty}$$

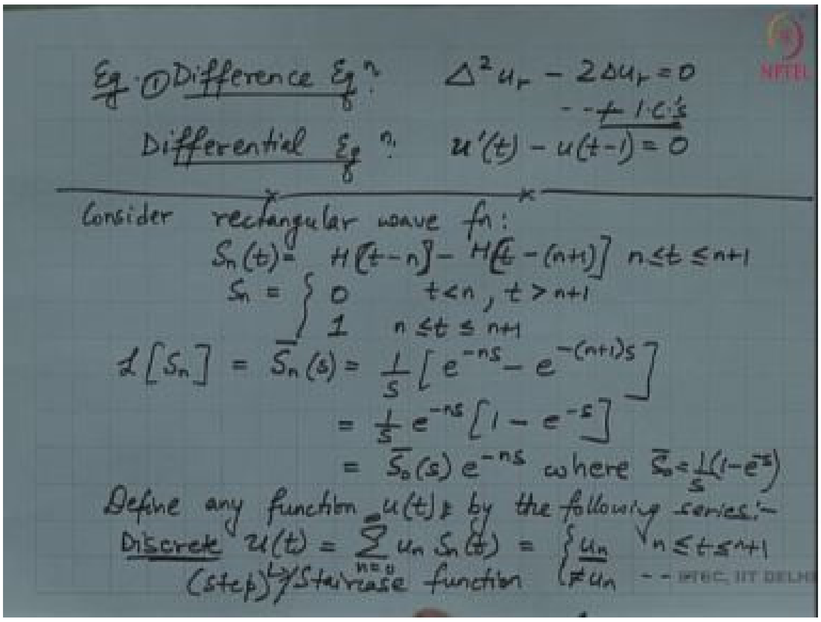
So, then I define, my operator delta by

$$\Delta u_r = u_{r+1} - u_r \tag{1}$$

$$\Delta^2 u_r = \Delta\left[\Delta u_r\right]$$
$$= \Delta\left[u_{n+1} - u_k\right]$$

We can define delta cube, delta 4 and so on.

$$= \Delta u_{r+1} - \Delta u_r = (u_{r+2} - u_{r+1}) - (u_{r+1} - u_r)$$
$$= u_{r+2} - 2u_{r+1} + u_r$$




So, then a typical difference a typical difference equation, let me show you a typical difference equation:

$$\Delta^2 u_r - 2\Delta u_r = 0$$

we have notice what is the difference between the difference and a differential equation. In a differential equation we take the proper derivative of the variable, the unknown variable u, with respect to the independent variable t. So, this is let us say equation of this form. So, this is the delay differential equation, we see that this is slightly different than equation that I am just referring, which is the difference equation.

$$u'(t) - u(t-1) = 0$$

So, to move ahead, to move ahead let me just introduce, some results. So, consider a rectangular wave function. So, what I am saying is consider this function

$$S_n(t) = H(t-n) - H(t-(n+1)) \quad n \leqslant t \leqslant n+1$$

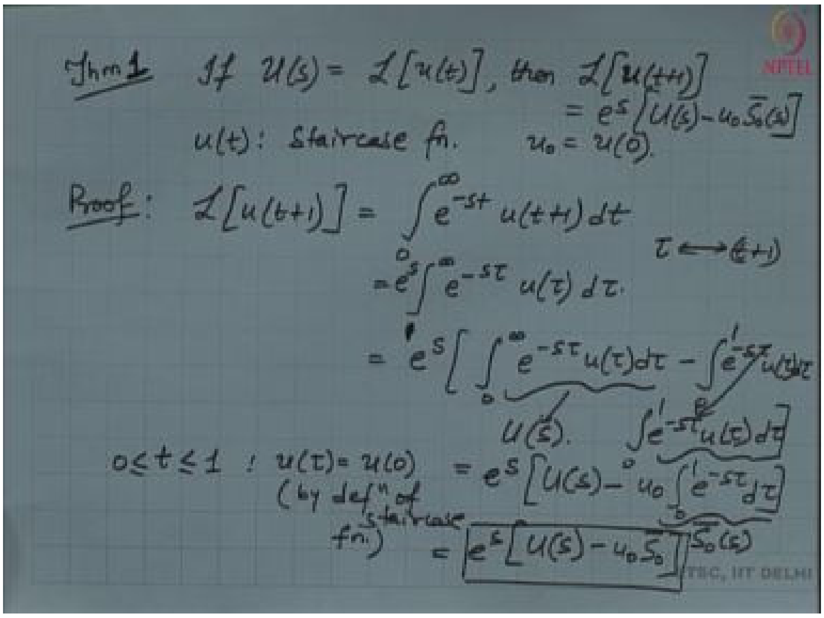
$$S_n = \begin{cases} 0 & t < n, t > n+1 \\ 1 & n \leqslant t \leqslant n+1 \end{cases}$$

So, so we see that, if we were to find the Laplace transform of this function of Sn, we see the following:

$$\begin{aligned} L\left[S_n\right] = \overline{S_n}(s) &= \frac{1}{s}\left[e^{-ns} - e^{-(n+1)s}\right] \\ &= \frac{1}{s}e^{-ns}\left[1 - e^{-s}\right] \\ &= \overline{S_0}(s)e^{-ns} \text{ where } \tilde{s}_e = \underline{l}\left(1 - e^{-s}\right) \end{aligned}$$

So, then let me just define any function; let me just define any function of u(t) of by some series by the following series, in the discrete sense; in the discrete sense I can define my u(t).So, this sort of a discrete definition of the function, where it attains a specific constant value is we call this sort of a function as a staircase function. So, this function gets a particular value, which is the constant value in each of the intervals :

$$u(t) = \sum_{n=0}^{\infty} u_n S_n(t) = \begin{cases} u_n & n \leqslant t \leq n+1 \\ z u_n & \end{cases}$$




So, then let me just provide you with one one theorem, before we are going to end today's discussion. So, the theorem says, I am going to denote this theorem by theorem 1. So, it says : Theorem 1:

$$\text{If } u(s) = L[u(t)] \text{ , then } L[u(t+1)] = e^s \left(u(s) - u_0 \overline{S_0}(s)\right]$$
$$u(t) : \text{ starcase } fn \text{ .} \quad u_0 = u(0)$$

So, let us look at the proof of this result, Proof:

$$L[u(t+1)] = \int_0^{\infty} e^{-st} u(t+1) dt$$

$$= e^s \int_0^{\infty} e^{-s\tau} u(\tau) d\tau$$

Now, I can rewrite this integral as

$$e^s \left[ \int_0^{\infty} e^{-s\tau} u(\tau) d\tau - \int_{e^{-s}}^{1} e^{-sx} u(\tau) dT \right]$$

$$0 \leqslant t \leqslant 1 : \quad u(\tau) = u(0)$$

Look at this second integral, the second integral is just a finite integral and in this finite integral between from t from 0 to 1:

$$= e^s \left[ u(s) - u_0 \int_0^1 e^{-s\tau} d\tau \right]$$

$$= e^s \left[u(s) - u_0\overline{S}_0\right]$$

So, this result is going to be useful in our further evaluation of further expressions and to solve difference equations and differential-difference equation and we will see in the next lecture, how to use these this result that I have just described along with some other results that were discussed in this lecture, to solve difference equation. And then we are going to look at some specific applications in some streams of electrical engineering, where we apply Laplace transform.

Thank you very much.