**Integral Transform and Their Applications**

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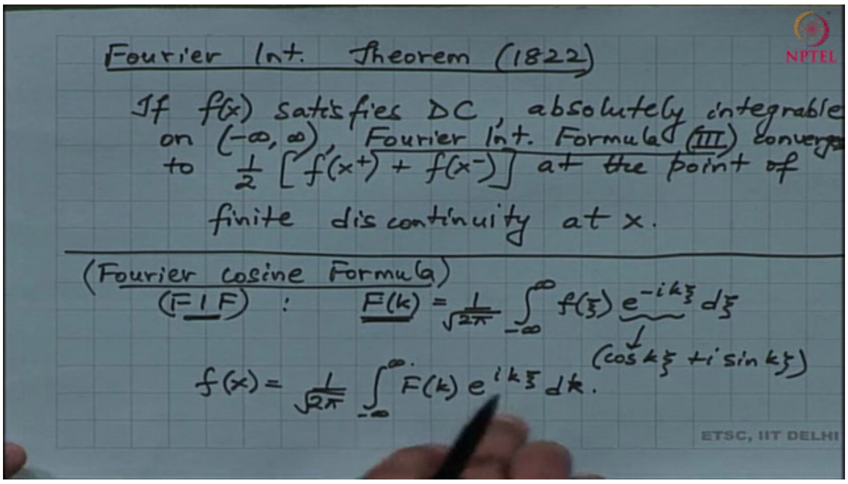
**Lecture-01**
**Introduction to Fourier Transforms Part 2**




So then from Fourier integral formula I can always find Fourier cosine formula. What is Fourier cosine formula? So, notice again my Fourier integral formula denoted by :

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) e^{-ik\zeta} d\zeta$$

We see that our this exponential is actually a complex number, this is given by $\cos(k\zeta) + i\sin(k\zeta)$. Now, my function

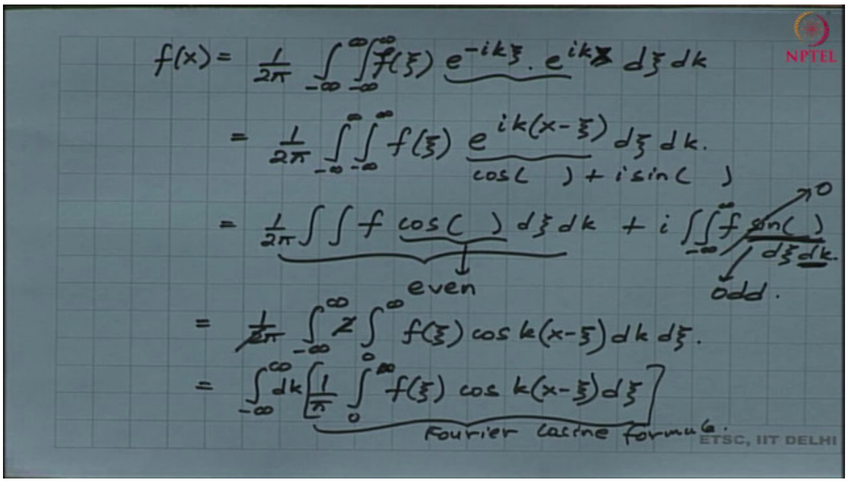
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ik\zeta} dk$$

so that is what we have seen. Combining these two, I am going to derive this Fourier cosine formula here,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\zeta) e^{-ik\zeta} \cdot e^{ikx} d\zeta dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\zeta) e^{ik(x-\zeta)} d\zeta dk \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\zeta) \cos(k(x-\zeta)) d\zeta dk + i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\zeta) \sin(k(x-\zeta)) d\zeta dk \end{aligned}$$

Notice that sin is an odd function. So, when I integrate sin over this variable k dk, when I integrate this sin value, it disappears and this integral is going to give me 0. So, I am only left with the cosine integral. Now, also the cosine function is an even function. So,

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2 \int_{0}^{\infty} f(\zeta) \cos(k(x-\zeta)) d\zeta dk \\ &= \int_{-\infty}^{\infty} dk \left[ \frac{1}{\pi} \int_{0}^{\infty} f(\zeta) \cos(k(x-\zeta)) d\zeta \right] \end{aligned}$$




So, why I have done that is because notice that this is my Fourier cosine formula. So, I have used the fact that cosine is an even function.

Similarly I have the so called Fourier sine formula, which I denote it as:

$$F_s(k) = \frac{1}{\pi} \int_0^{\infty} f(\zeta) \sin k(x-\zeta) d\zeta$$

I am going to distinguish between the sin as $F_s(k)$. So, s denotes sine and if I have c, it is cosine. So, that is the way we define our Fourier cosine and sine formula. Now, when to use the Fourier sine formula and when to use the Fourier cosine formula? If I have that f is an even function, then the natural choice is to use the Fourier cosine formula and in that case

$$F_c = \frac{2}{\pi} \int_0^{\infty} f(\zeta) \cos k(\zeta) d\zeta$$

So, this is the case when I have $f(x) = f(-x)$, when f is even.

And of course, similarly if f is odd, I can always use my Fourier sine formula as notice that the 2 is coming out, why 2 is coming out? Because of the fact that we have expanded our sine and cosine formula using our summation rule; so that has brought down this factor here. So, similarly

$$F_s = \frac{2}{\pi} \int_0^{\infty} f(\zeta) \sin k(\zeta) d\zeta.$$

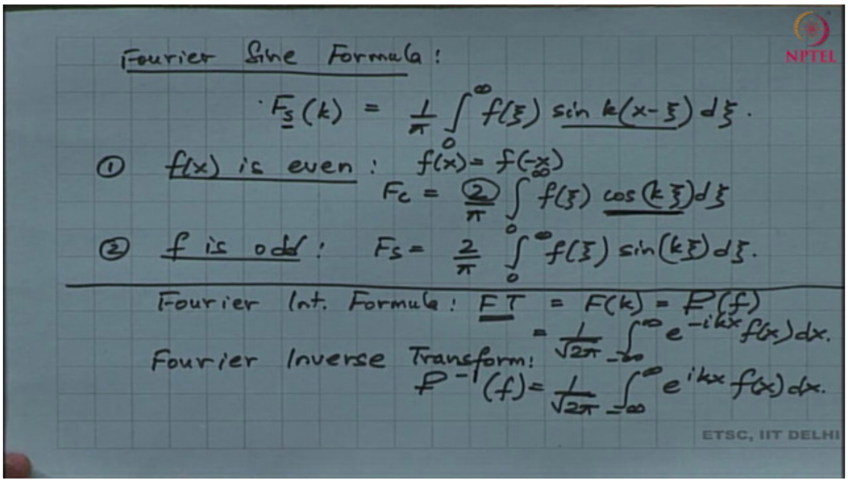
So, let us now recap a little bit. So, the Fourier integral formula, I also call it as Fourier transform denoted by F T, which is given to be

$$F(k) = \mathcal{F}(f)$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx$$

And then of course, we have Fourier cosine formulas, Fourier sine formulas. Most important thing is the Fourier inverse, what is the inverse formula for, or the Fourier inverse transform?

The inverse transform is:

$$\mathcal{F}^{-1}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} f(x) dx$$




So, I just want to recap that whenever we have a problem which is non-periodic, we are going to use Fourier transforms. Whenever we have functions which are non-periodic, but it is defined on a half domain and it is odd, then we are going to use the Fourier sine series. And whenever the function is non-periodic and also the function is even, it is natural to use the Fourier cosine series. So, please remember all these facts when we are going to do some application problems.

Now, before I move ahead, I just want to highlight there is another notation which is widely used by people in electrical engineering. So, Fourier transforms are mostly in electrical engineering, in definition of signals. And in that case, my variable x in the Fourier transform is going to be replaced by another variable t. Now, as we will see of course in the application t will denote time, but let us keep it general, t is another variable that is replaced by a from x.

And similarly my variable k in the transformed plane is replaced by $\omega = 2\pi\nu$. So, what is this? $\omega$ is sometimes also called the angular frequency and k in our transformed plane is called as the wave number. So, I am talking about all these concepts, because we are going to introduce all these notions, when we do some problems.

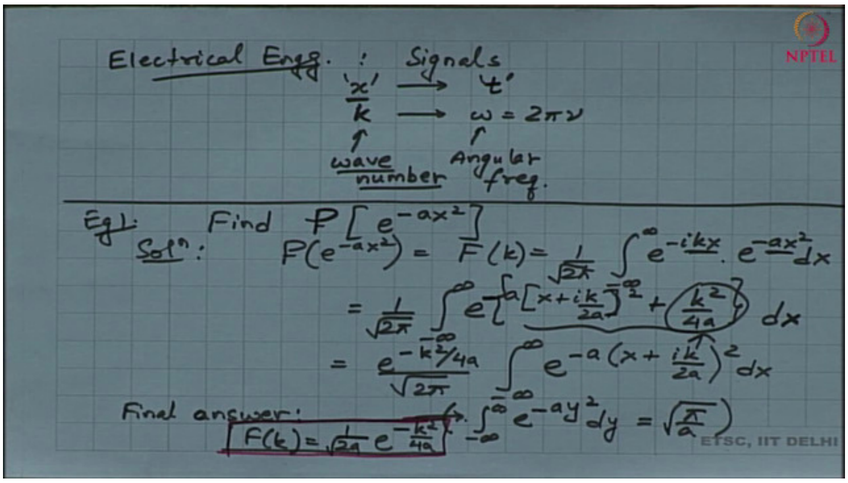
So, let us move ahead. So, here is one quick example to show you how Fourier transforms are used. So, the example is find $\mathcal{F}\left[e^{-ax^2}\right]$.

So, its a very straight forward application of Fourier transform.

$$\mathcal{F}\left[e^{-ax^2}\right] = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} e^{-ax^2} dx$$

Now, we have to now integrate this integral which is an infinite integral, so that becomes the way to do this integral is we are going to complete the squares. So, what do I mean by that? Please note that I can write the factor on top of this exponential as follows:

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[a\left(x+\frac{ik}{2a}\right)^2 + \frac{k^2}{4a}\right]} dx \\ &= \frac{e^{-k^2/4a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a\left(x+\frac{ik}{2a}\right)^2} dx \end{aligned}$$




Now, there is a very standard theorem in calculus which tells us that if I have an infinite integral of this factor $\int_{-\infty}^{\infty} e^{-ay^2} dy = \sqrt{\pi/a}$. So, I am going to use this result it is available quite common in calculus books and let us use this result to come to the final answer. So, the final answer is that $F(k) = \frac{1}{\sqrt{2a}} e^{-k^2/4a}$. So, let me just use a red pen to highlight the answer here. So, let us do some more examples.

So, I have to find the Fourier transform of this function,

$$\mathcal{F}\left[\left(1 - \frac{|x|}{a}\right) H\left(1 - \frac{|x|}{a}\right)\right]$$

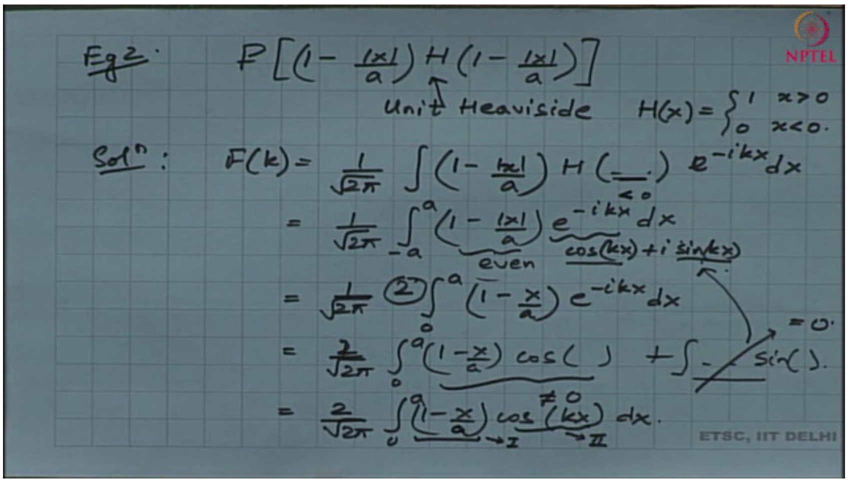
Now, what is this function H? The H is called the Heaviside function right. So, if I have $H(x)$, it is defined as 1 if x is positive and 0 if x is negative. So, this is a unit, let me call it as a unit Heaviside, function because it gives you either 1 or 0 ok. So, then my Fourier transform of this function will be again

$$\mathcal{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(1 - \frac{|x|}{a}\right) H\left(1 - \frac{|x|}{a}\right) e^{-ikx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} \left(1 - \frac{|x|}{a}\right) e^{-ikx} dx$$

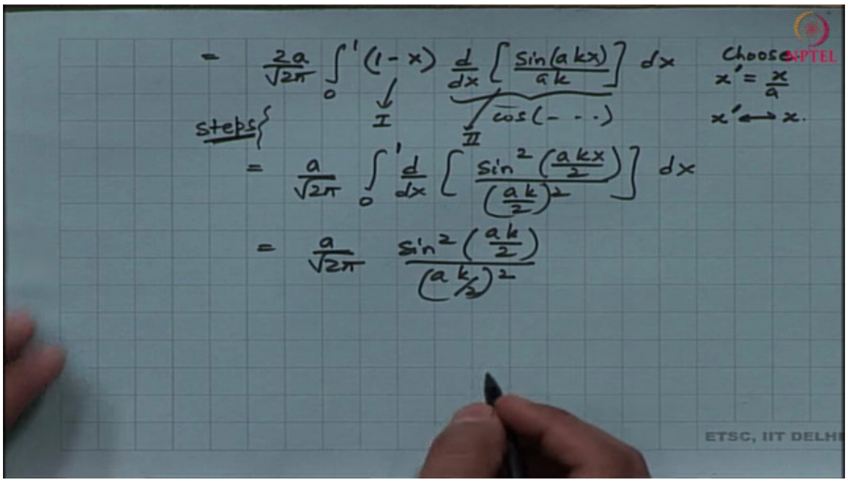
, notice that Heaviside function vanishes; if the argument is less than 0, it vanishes when the argument is less than 0, which means that my Fourier integral reduces to this finite integral. Why does this vanishes transforms into this finite integral? Because only in the finite integral this Heaviside function is takes the value 1, otherwise this Heaviside function takes the value 0. So, then you see that this function is an even function, because if I replace x by minus x, the value does not change. So, I can always change this integral into a new integral,

$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_{0}^{a} \left(1 - \frac{x}{a}\right) e^{-ikx} dx$$
$$= \frac{1}{\sqrt{2\pi}}\, 2 \int_{0}^{a} \left(1 - \frac{x}{a}\right) \cos(kx) dx + \int_{0}^{a} \left(1 - \frac{x}{a}\right) \sin(kx) dx$$

this is 2 times integral from 0 to a 1 minus x by a e to the power i k x d x. I have used the fact that the function the earlier function was an even function, so that has brought in this factor of 2 ok. So, then you can see that this is well what next to be done is the fact that, well we see that this is also cos of k x plus i times sin of k x right.




Well it actually makes sense, if we see this case here right. So, if I were to break this integral into cosine and the sine case and my second this function is an even function. So, the product of even function with an even function will be an even function and the product of an even function with an odd function here sin is an odd function will be an odd function. So, an odd function over an interval from -a to a is going to give me 0. So, all I am going to get is $\frac{1}{\sqrt{2\pi}}\ 2\int_0^a \left(1-\frac{x}{a}\right)\cos(kx)dx$. So, then what to do next we are going to use integration by parts. So, I am going to take $\left(1-\frac{x}{a}\right)$ as my first function and $\cos(kx)$ as my second function.

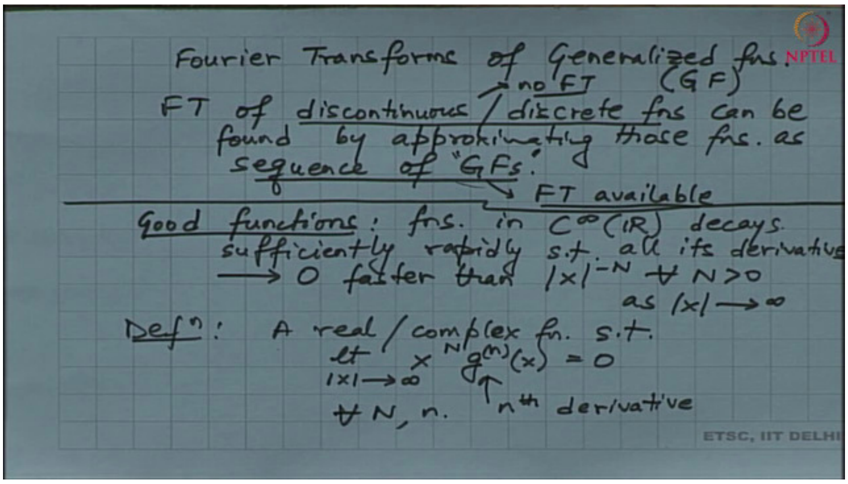



So, let me write it down. So, I have the following

$$
\begin{aligned}
&= \frac{2a}{\sqrt{2\pi}}\int_0^1 (1-x)\frac{d}{dx}\left[\frac{\sin(akx)}{ak}\right]dx \\
&= \frac{a}{\sqrt{2\pi}}\int_0^1 \frac{d}{dx}\left[\frac{\sin^2(akx/2)}{(ak/2)^2}\right]dx \\
&= \frac{a}{\sqrt{2\pi}}\frac{\sin^2(ak/2)}{(ak/2)^2}
\end{aligned}
$$

What I have done here is, I have used a new variable $x'$ which is $\frac{x}{a}$. So, choose my new variable $x'$ as $\frac{x}{a}$ and again I replace ,since $x'$ is a dummy variable I replace $x'$ by x again. I am leaving all the calculations here, there are some steps which I request the students to see what they are integration by parts. Ok, so far we have seen some examples of the use of Fourier transforms.

So, let me now introduce the Fourier transform. The problem comes how to evaluate Fourier transform of functions in which we have some difficulties right. So, specifically I am going to introduce Fourier transforms of generalized functions. So, what are these generalized functions? So, Fourier transforms of discontinuous or discrete functions can be found by approximating those functions as sequence of let me call this generalized function as G F, sequence of Generalized Functions. So, what happens is that most of the times this discontinuous or discrete functions, we cannot evaluate Fourier transforms right. So, it is better to approximate these continuous or discrete functions into the sequence of the so called generalized functions that I am going to introduce now. And the Fourier transforms of these generalized functions are available right. So, next I am going to show that if I am able to figure out the Fourier transform of the sequence of generalized functions, I will show that that corresponds to the Fourier transform of the corresponding discrete or discontinuous function right. Now, so the long story short if I am not able to evaluate the Fourier transform of a certain function, all I need to do is to represent that function into a sequence of so called generalized functions, for which Fourier transforms are available right. Now, I am going to introduce some more notions. One of the other notions that I introduce is the so called good functions. So, what are good functions? So, good functions are those functions which are infinitely differentiable over the real line, I denote it by $C^{\infty}(R)$. And it decays sufficiently, it decays sufficiently rapidly such that all its derivatives; all its derivatives decays to 0 faster than $x^{-N}$ that is for all integers N positive as my variable X goes to infinity.

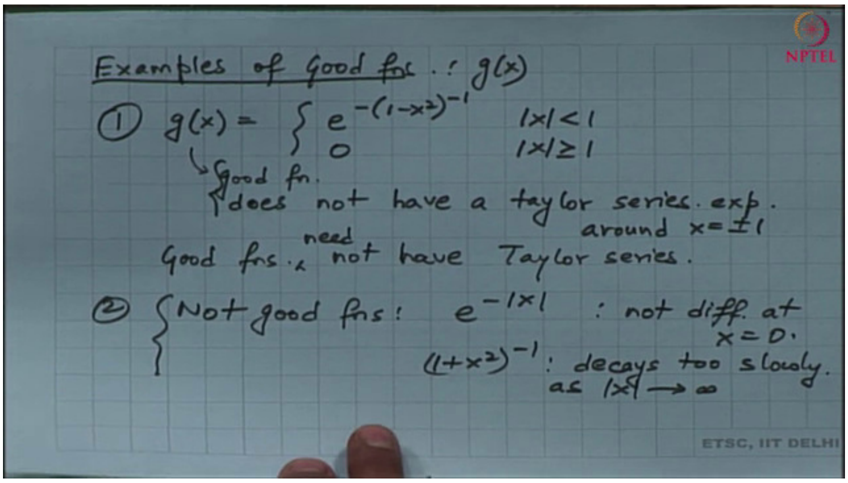



Now, to give you the definition in short, good functions are those real or complex valued functions such that I have this following limit goes to 0. So, limit x tending to minus or plus infinity $x^N g^{(n)}(x) = 0$. So, I am taking, here I am taking the nth derivative right. So, the nth derivative decays faster than $x^{-N}$, for all N and n.

So, then what do I have is that let me just give you some examples of good functions right. So, one of the examples is well this function let me denote it by this function $g(x)$. So, here one example of a good function which is to be given by

$$g(x) = e^{-(1-x^2)^{-1}}, \qquad |x| < 1$$

and 0 otherwise. We see that in this case, this is a good function if we use the definition, but it does not have a Taylor series expansion around x equals plus minus 1 right. For the other cases its easy to see the Taylor series exist, but around $+_-1$, we see that in for this for good function the Taylor series does not exist. So, what is the conclusion, the conclusion is that good functions need not well, good functions need not have Taylor series expansion right. So, that is the slightly against the intuition that we have. Then I must give you an example of functions which are not good function. So, one of the cases is this $e^{-|x|}$, why because it is not differentiable at x equals 0. Another case of not good function is $(1 + x^2)^{-1}$, why because it decays too slowly, then what the definition tells us as x goes to $+_-\infty$. So, at least we know what are good functions and which functions are not good functions.




So, then one of the properties is if I have a function which is good a good function, then the function g which is defined as the integral of good function, the integral of good function is also a good function if and only if this integral of this good function exists right. So, if you can evaluate the integral this is finite well, the integral of a good function is also good function right. Well another property is good functions are absolutely continuous. So, I am not going to go into the definition of what is absolutely continuous that could be found in regular calculus text well, what I want to highlight is it is much stronger than your continuity; than your continuity or uniform continuity scenario right. So, it is absolutely continuous here. Then I want to introduce another function another definition called.