Integral Transforms and Their Applications
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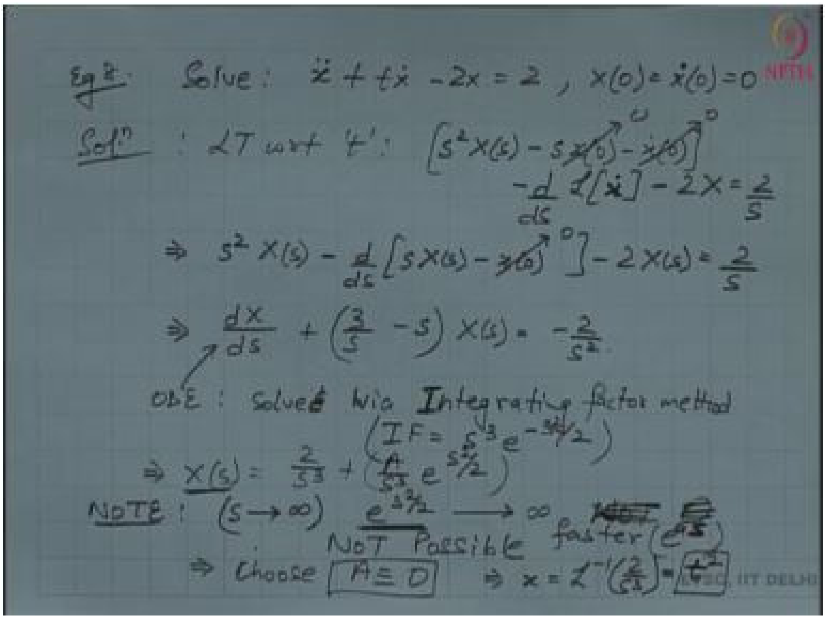
Lecture –06
Applications of Laplace Transforms Part 3



Let us look at another situation I need to solve this:

Example 8:

$$\text{Solve:} \quad \ddot{x} + t\dot{x} - 2x = 2, x(0) = \dot{x}(0) = 0$$

So, when I simplify this I get the following :

Solution:

$$LT \text{ w.r.t } t^{:(s^2x(s) - sx(0) - x(0)]}$$
$$-\tfrac{d}{ds}L[x] - 2x = \tfrac{2}{s}$$

$$s^2x(s) - [\left[y\frac{d}{ds}sx(s) - x(0)\right] - 2x(s) = \frac{2}{s}$$

$$\frac{dx}{ds} + \left(\frac{3}{s} - s\right)x(s) = -\frac{2}{s^2}$$

Notice that this is an ODE and this can be solved via the integrating factor method integrating factor method. So, I am assuming that students already have some brief idea about how to solve this ODE via integrating factor method I see that I am going to choose my integrating factor IF as:

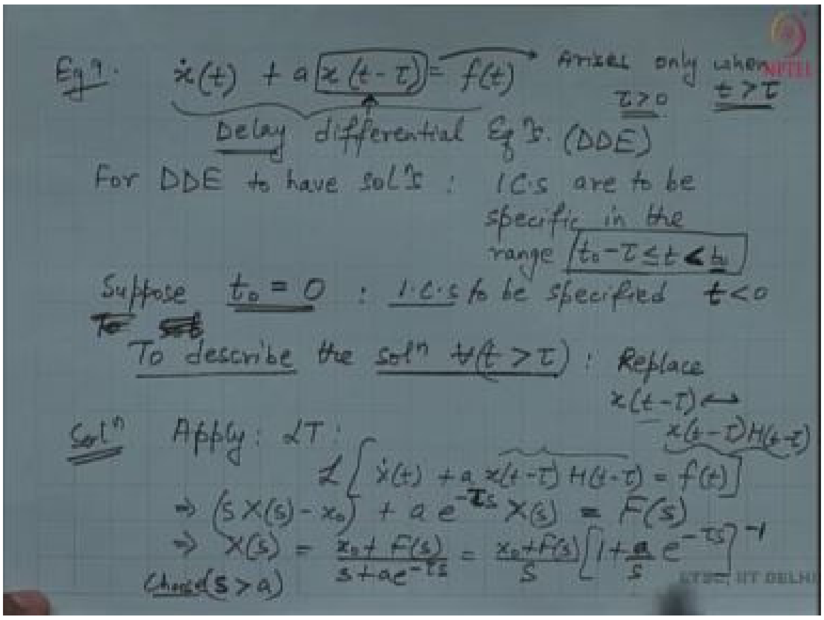
$$IF = s^3e^{\frac{s^2}{2}}$$

$$x(s) = \frac{2}{s^3} + \left(\frac{A}{s^3}e^{\frac{s^2}{2}}\right)$$

$$\text{Note: } (s \to \infty) \quad e^{s^2/2} \to \infty$$
$$\Rightarrow \text{ Choose } [A \equiv 0]$$

It blows up and this is not of the exponential; so, this blows up faster, I would say this is blows up faster than e to the power s right or as. So, it blows up faster than the exponential order. So, which means that this sort of a term should not be there in the Laplace transform if we; if that this function is a valid Laplace transform of some known function right. So, we are going to choose; so, this is not possible due to the reasons that x of s is the Laplace transform of some variable small x. So, I am going to choose my A; the constant A to be identically 0 so, that this term vanishes .

$$x = L^{-1}\left(\frac{2}{s^3}\right) = t^2$$



So, then let us look at few more cases, I have another example where I am given an ODE:

Example 9:

$$\dot{x}(t) + ax(t - \tau) = f(t)$$
$$\text{Arises only when } t > 0 \quad , t > \tau$$

Notice that this is not a simple ODE, this is also called Delay Differential Equation; equations of this form are also called Delay Differential Equations notice that here the solution in this term is lagging by so called factor tau and I am going to choose my tau to be positive here.

So, just give you a brief introduction to delay differential equations. So, for DDE to have solutions, we must have the initial conditions are to be specified in the range :

For DDE to have solutions ICS are to be
specific in the
range $(t_0 - \tau \le t < t_0)$

Suppose $t_0 = 0$: ICS to be specified $t < 0$

$$\forall (t > \tau): \text{ Replace}$$

To describe the solution: $$x(t-\tau) \leftrightarrow$$
$$x(t-\tau)H(t-\tau)$$

Solution: So, that hopefully describes what sort of an equation we are solving. So, now from a delay equation I have changed it to a regular equation using this heavy side function. So, the solution is as follows. So, again we apply my Laplace transform to this variable:

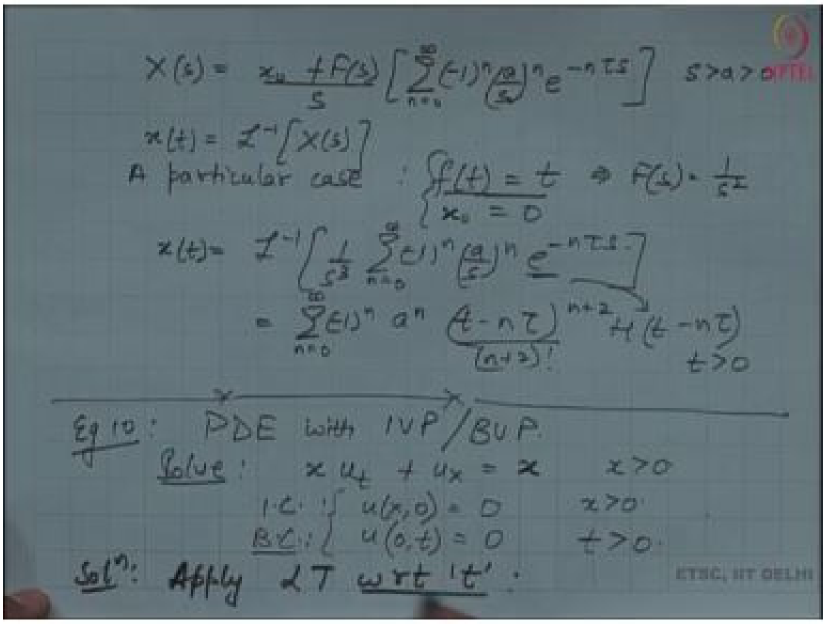
Apply LT:

$$L\left[x(t) + ax(t-\tau)H(t-\tau) = f(t)\right]$$

$$\Rightarrow (sx(s) - x_0) + ae^{-\tau s}x_{(s)} = F(s)$$

$$x(s) = \frac{x_0 + f(s)}{s + ae^{-t_s}} = \frac{x_0 + f(s)}{s}\left[1 + \frac{a}{s}e^{-\tau s}\right]^{-1}$$

$$\text{choose: } (s > a)$$



$$x(s) = \frac{x_0 + f(s)}{s}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n e^{-n\tau s}\right] \quad s > a > 0$$

$$x(t) = \mathcal{L}^{-1}[x(s)]$$

$$\text{A particular case: } f(t) = t, x_0 = 0 \Leftrightarrow F(s) \cdot \frac{1}{s^2}$$

$$x(t) = L^{-1}\left[\frac{1}{s^3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{s}\right)^n e^{-n\tau s}\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n a^n \frac{(t-n\tau)^{n+2}}{(n+2)!}H(t-n\tau), t > 0$$

So, that completes the solution to this delay differential equation let us now look at one more equation I am going to look a PDE; a PDE with initial value so, it is an initial value as well as a boundary value problem

PDE with IVP/BUP

Solve:

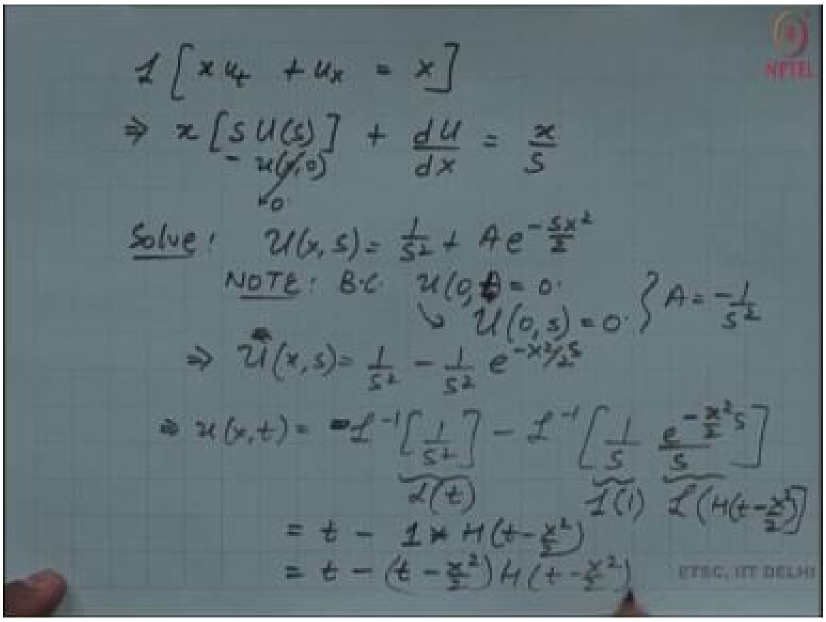
$$xu_t + u_x = x \quad , x > 0$$

$$\begin{array}{ll} \text{IC:} & u(x,0) = 0 \\ x > 0 & \\ BC : u(0,t) = 0 \quad t > 0 & \end{array}$$

So, when I apply the Laplace transform so, let us start by applying the Laplace transform with respect to one of the variable

Solution:

Applying Laplace Transform:



$$L\left[xu_t + u_x = x\right]$$

$$x\left[\begin{array}{c} su(s) \\ -u(x,0) \end{array}\right] + \frac{du}{dx} = \frac{x}{s}$$

Solve:

$$\mathcal{U}(x,s) = \frac{1}{s^2} + Ae^{-\frac{sx^2}{2}}$$

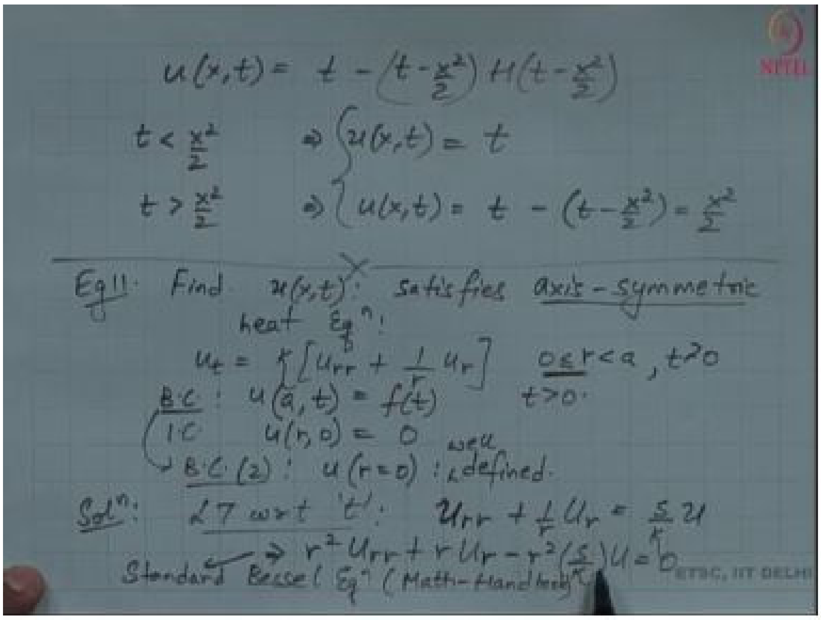
$$\text{BC:} \quad \left.\begin{array}{c} u(0,A) = 0 \\ u(0,s) = 0 \end{array}\right\} A = \frac{-1}{s^2}$$

$$u(x,s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-x^2/2^s}$$

$$u(x,t) = -L^{-1}\left[\frac{1}{s^2}\right] - L^{-1}\left[\frac{1}{s}\frac{e^{-\frac{x^2}{2}}s}{s}\right]$$

$$= t - 1 \times H\left(t - \frac{x^2}{2}\right)$$

$$= t - \left(t - \frac{x^2}{2}\right)H\left(t - \frac{x^2}{2}\right)$$



$$u(x,t) = t - \left(t - \frac{x^2}{2}\right) H\left(t - \frac{x^2}{2}\right)$$

$$\begin{array}{ll} t < \frac{x^2}{2} & \Leftrightarrow \int u(x,t) = t \\ t > \frac{x^2}{2} & \Rightarrow \Big\{ u(x,t) = t - \left(t - \frac{x^2}{2}\right) = \frac{x^2}{2} \end{array}$$

So, then let us come to another example, let us say example 11 that is the number that I am following. So, I have to find the solution to this PDE;

Example 11:

Find u(x,t) satisfies axis-symmetric heart equation:

$$u_t = k\left[u_{rr} + \frac{1}{r}u_r\right] \quad 0 \leq r < a, t > 0$$

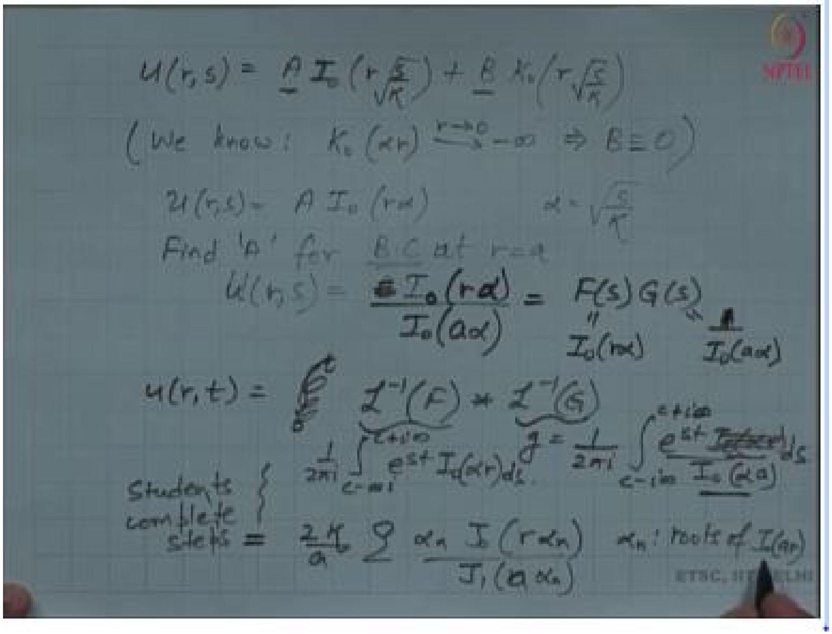
$$\begin{array}{ll} \text{B.C.} & u(a,t) = f(t) \quad t > 0 \\ \text{IC} & u(r,0) = 0 \end{array}$$

Solution:

Laplace Transform w.r.t t:

$$u_{rr} + \frac{1}{r}u_r = \frac{s}{k}u$$

So this, why I wrote this in this form because this is the standard Bessels equation the standard Bessels equation. So, please look at the mathematical hand book math handbook that I have prescribed to see that this is indeed the case. So, this is a Bessels equation and the answer to this equation is the standard Bessels function. So, I am going to directly write the answer to this ODE.



$$u(r,s) = \underline{A} I_0 \left(r\sqrt{\frac{s}{k}} \right) + \underline{B} K_0 \left(r\sqrt{\frac{s}{k}} \right)$$

$$\text{We know: } k_0\left(\alpha r\right) \rightarrow^{r \rightarrow 0} -\infty \rightarrow B \equiv 0$$

$$u(r,s) = A I_0(r\alpha), \alpha = \sqrt{\frac{s}{k}}$$

Find A for B.C. at r=a,

$$u(r,s) = \frac{I_0\left(r_d\right)}{I_0\left(a_\alpha\right)} = \quad F(s)G(s)$$

$$u(r,t) = L^{-1}(F) * L^{-1}(G)$$

So, that completes we see that we can easily find these contours using the poles of these modified Bessels function or the roots of this modified Bessels functions right. And we can write the answer right away to come to the following expression:

$$= \frac{2k}{a} \sum \frac{J_0 r\alpha_n}{J_1\left(a\alpha_n\right)}$$

So, at this point I am going to end this discussion and in the next discussion, I am going to talk about more about Laplace transform and its applications and look at some more useful theorems in which we using which we are going to evaluate some more real life problems.

Thank you very much.