Integral Transforms and Their Applications

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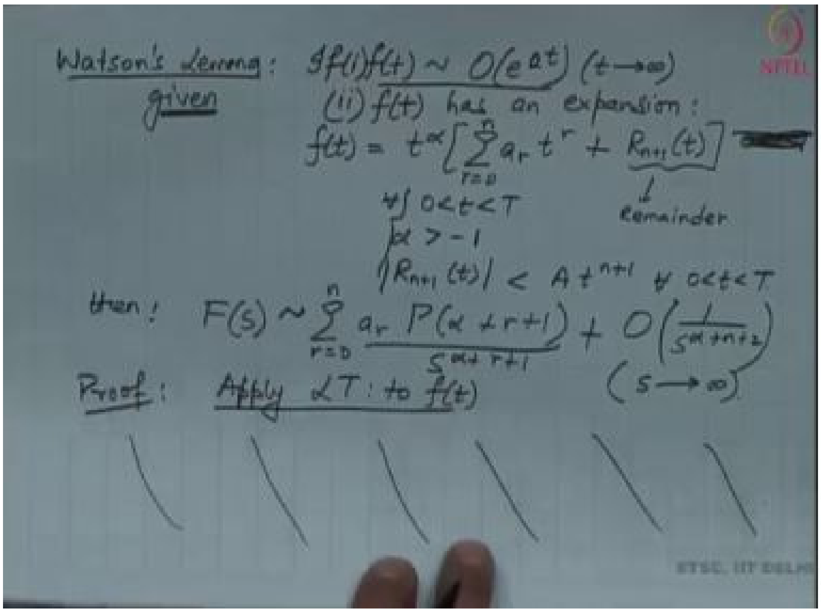
Lecture –06
Applications of Laplace Transforms Part 2



Moving on, I am going to provide some example for or application examples, where I am going to show how to use all these nice results like the Tauberian theorems the Watsons lemma. And also some of the properties of Laplace transform to evaluate certain ODEs and PDEs ok; so, relevant ODEs and PDEs.

Watson lemma:

Given:

If (1) $f(t) \sim O\left(e^{at}\right)(t \to \infty)$

(2) f(t) has an expansion:

$$f(t) = t^{\alpha}\left[\sum_{r=0}^{\infty} a_r t^r + R_{n+1}(t)\right]$$

$$\forall 0 < t < T$$

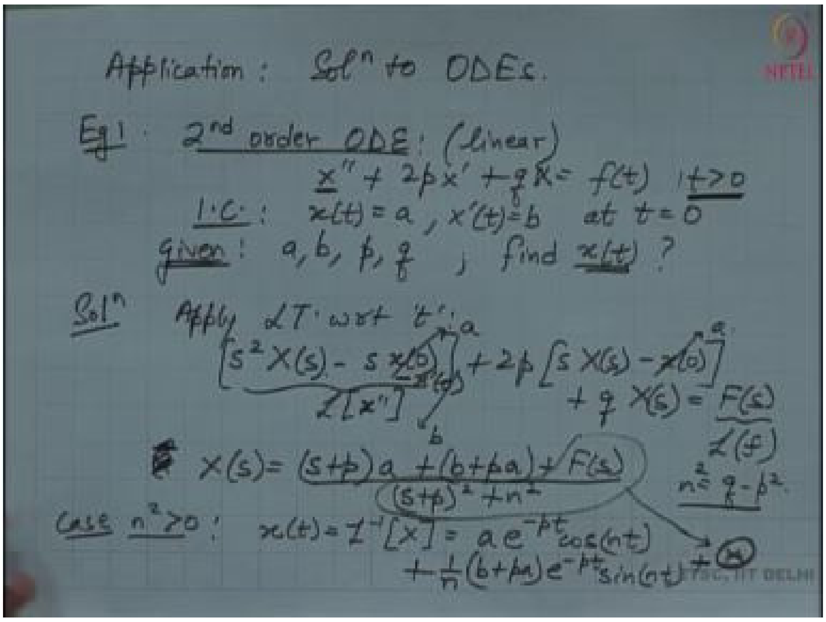
$$| \begin{array}{l} \alpha > -1 \\ R_{n+1}(t)| < At^{n+1}, \forall 0 < t < T \end{array}$$

then:

$$F(s) \sim \sum_{r=0}^{n} \frac{a_r P(\alpha + r + 1)}{s^{\alpha+r+1}} + O\left(\frac{1}{s^{\alpha+n+2}}\right)(s \to \infty)$$

Proof:

Apply LT to f(t):



So, to move on, I am going to start with solution to ODEs. So, let us look at one example. So, let us look at a 2nd order ODE; 2nd order ODE where I have so, in fact, a second order linear ODE ok. So, I am given this ODE :

$$\frac{2^{nd} \text{ order } ODE: \text{ linear}}{x'' + 2px' + qx = f(t), t > 0}$$

IC: $x(t) = a, x'(t) = b$ at $t = 0$

given: a, b, p, q , find $x(t)$?

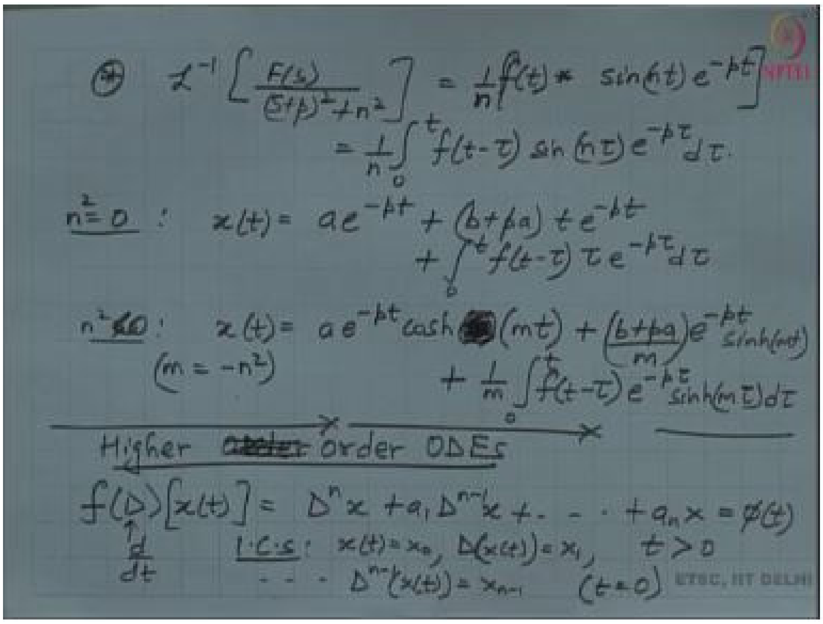
Now, notice that my variable my independent variable here is the t and the t is non negative. So, when we have a dependent variable which depends on the independent variable and the independent variable is nonnegative the natural choice of the transform is to use Laplace transform right. So, in that case we apply Laplace transform, with respect to t here.

Apply Laplace Transform wrt t:

$$\left[s^2x(s) - sx(0)\right] + 2p[sx(s) - x(0)] + qx(s) = F(s)$$

$$x(s) = \frac{(s+p)a + (b+pa) + (f(s))}{(s+p)^2 + n^2} \quad , n^2 = q - p^2$$

$$\text{Case } n^2 > 0 \text{ ; } x(t) = L^{-1}[x] = ae^{-pt}\cos(nt)$$
$$+\frac{1}{n}(b+pa)e^{-pt}\sin(nt)$$



$$L^{-1}\left[\frac{F(s)}{(s+p)^2+n^2}\right] = \frac{1}{n} f(t) * \sin(nt)e^{-pt}\Big]$$

$$= \frac{1}{n}\int_0^t f(t-\tau)\sin(n\tau)e^{-p\tau}d\tau$$

$$n^2 = 0: \quad x(t) = ae^{-pt} + (b+pa)e^{-pt}$$
$$+\textstyle\int_0^t f(t-\tau)\tau e^{-tpt}d\tau$$

$$n^2 < 0: \quad x(t) = ae^{-pt}\cosh(mt) + \left(\frac{b+pa}{m}\right)e^{-pt}\sinh(mt) + \frac{1}{m}\int_0^t f(t-\tau)e^{-p\tau}\sinh(m\tau)d\tau$$

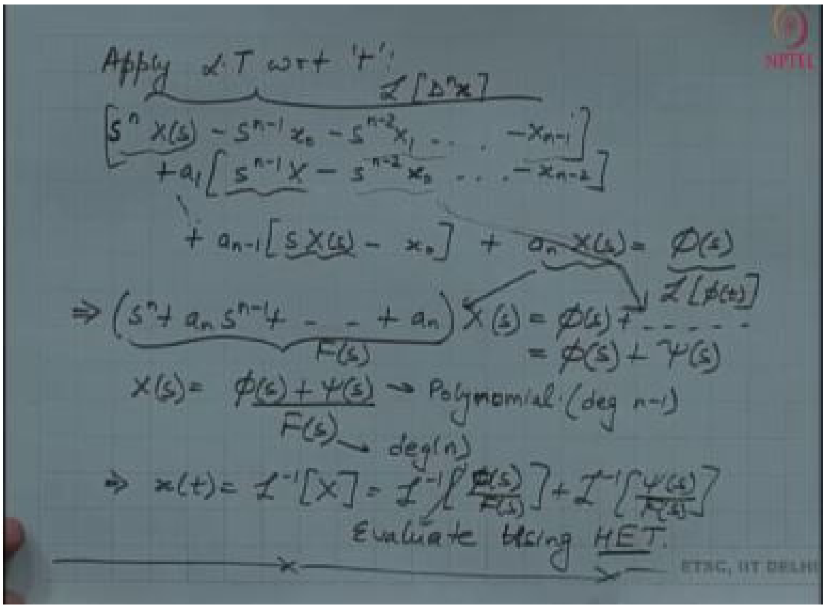
Now, suppose we look at the higher the situation of higher order ODEs:

Higher oredr ODES:

$$f(D)[x(t)] = D^n x + a_1 D^{n-1}x + \ldots + a_n x = \phi(t)$$

$$\text{ICS: } x(t) = x_0, D(x(t)) = x_1, \quad t > 0$$

$$\cdots\cdot D^{n-1}(x(t)) = x_{n-1} \quad (t=0)$$



Apply Laplace Transform wrt t:

$$s^n x(s) - s^{n-1}x_0 - s^{n-2}x_1 \ldots - x_{n-1} + a_1\left[s^{n-1}x - s^{n-2}x_0 \ldots - x_{n-2}\right] + a_{n-1}\left[sx(s) - x_0\right] + a_n x(s) = \varnothing(s)$$

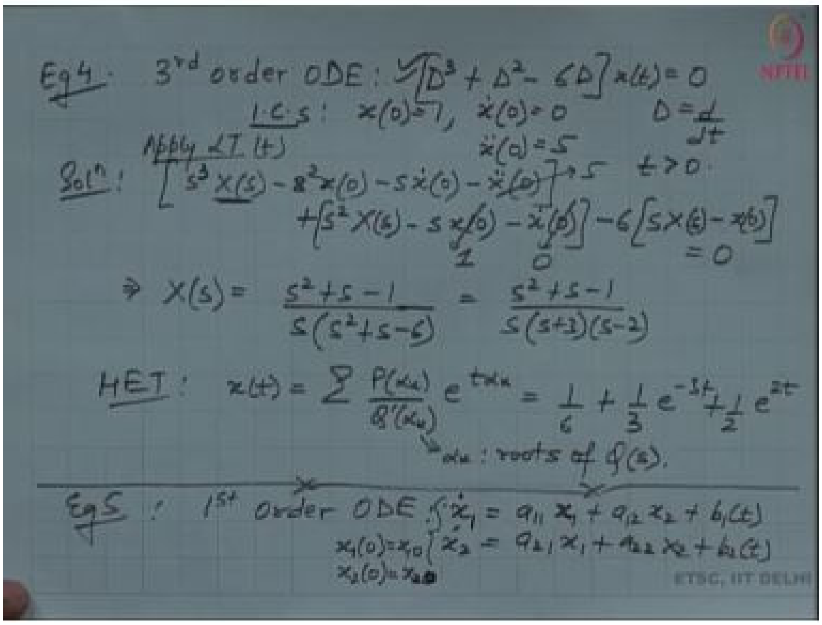
$$\left(s^n + a_n s^{n-4} + \cdots + a_n\right) x(s) = \phi(s) + \cdots = \phi(s) + \psi(s)$$

$$x(s) = \frac{\phi(s) + \psi(s)}{F(s)}$$

$$x(t) = L^{-1}[x] = L^{-1}\left[\frac{\phi(s)}{F(s)}\right] + L^{-1}\left[\frac{\psi(s)}{F(s)}\right]$$

Evaluate using HET:

now to evaluate this I use my heavy side expansion theorem evaluate using heavy side expansion theorem ok. So, and since all the properties of the heavy side expansion theorems are being satisfied in this case. So, that is how we approach to solve nth order ODEs.



So, let us now look at the scenario of 3rd order ODE. So, let us look at an example:

Example 4:

$$3^{rd} \text{ order } 0DE : \left[D^3 + D^2 - 6D\right]x(t)\big) = 0$$

$$IC: \begin{array}{ll} x(0) = 1, & \dot{x}(0) = 0 \\ & \ddot{x}(0) = 5 \end{array}$$

So, then when I apply the Laplace transform I get the following.:

Solution:

$$[s^3x(s) - s^2x(0) - 5\dot{x}(0) - \ddot{x}(0)] + \left[s^2x(s) - sx(0) - x(0)\right] - 6[sx(s) - x(0)] = 0$$

$$\Rightarrow x(s) = \frac{s^2 + s - 1}{s\,(s^2 + s - 6)} = \frac{s^2 + s - 1}{s(s+3)(s-2)}$$

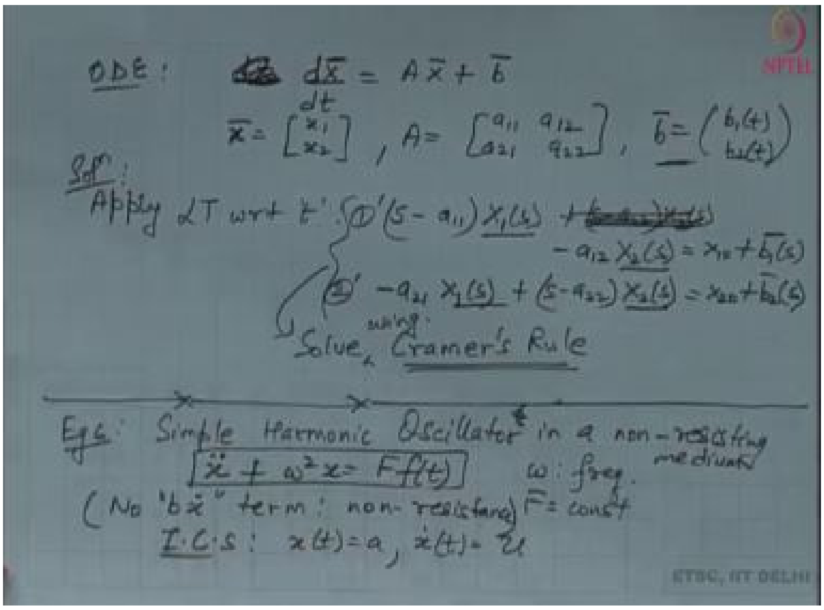
$$\text{HET:} \quad x(t) = \sum \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{t\alpha_k} = \frac{1}{6} + \frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}$$

Now, what if suppose we are given 2 ODEs or 3 ODEs and we have to solve simultaneously. So, let us look at a 1st order ODE and I am ask to find the solution this problem:

Example 5:

1st order ODE:

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + b_1(t)$$
$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + b_2(t)$$
$$x_1(0) = x_0$$
$$x_2(0) = x_2$$



ODE:

$$\text{ODE:} \quad \frac{d\overline{x}}{dt} = A\overline{x} + \overline{b}$$

$$\overline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{12} \end{bmatrix}, \quad \overline{b} = \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$$

Solution:

Apply Laplace Transform:

$$(1) = (s - a_{11})\, x_1(s) - a_{12}x_2(s) = x_{10} + \overline{b_1}(s)$$
$$(2) = -a_{21}x_1(s) + (s - a_{22})\, x_2(s) = x_{20} + \overline{b_2}(s)$$

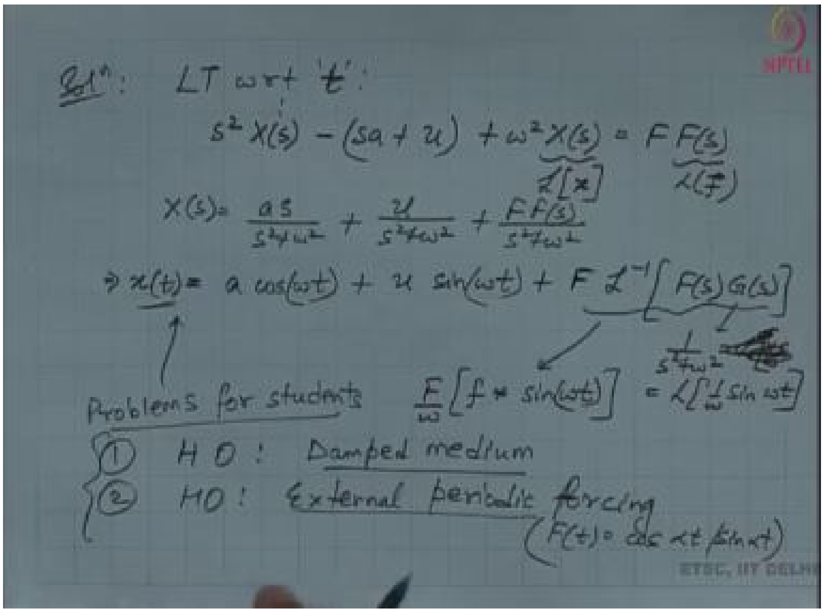
So, I leave the answer to the students because this can be directly evaluated by using Cramer's rule by the method of determinants this is a linear equation and then the suitable inverse can be taken if you know what are these right hand term coefficients b is ok. So, that is the solve using Cramer's rule. So, that is the end of this discussion.

Let us look at one example an example of the simple harmonic oscillator; simple harmonic oscillator in I am going to look at a simple case in a non resisting medium. So, what I mean by that is I have the following ODE:

Example 6:

$$\ddot{x} + \omega^2 x = Ff(t), \omega : frequency$$
$$IC : x(t) = a, x(t) = u$$

So, simple harmonic oscillator. So, if I were to apply, so, this is my equation and I am going to apply the Laplace transform on this equation.



Solution:

Apply Laplace transform:

$$s^2x(s) - (sa + u) + \omega^2x(s) = FF(s)$$

$$x(s) = \frac{as}{s^2 + w^2} + \frac{u}{s^2 + w^2} + \frac{FF(s)}{s^2 + w^2}$$
$$\Rightarrow x(t) = a\cos(\omega t) + u\sin(\omega t) + F\mathcal{L}^{-1}[F(s)G(s)]$$

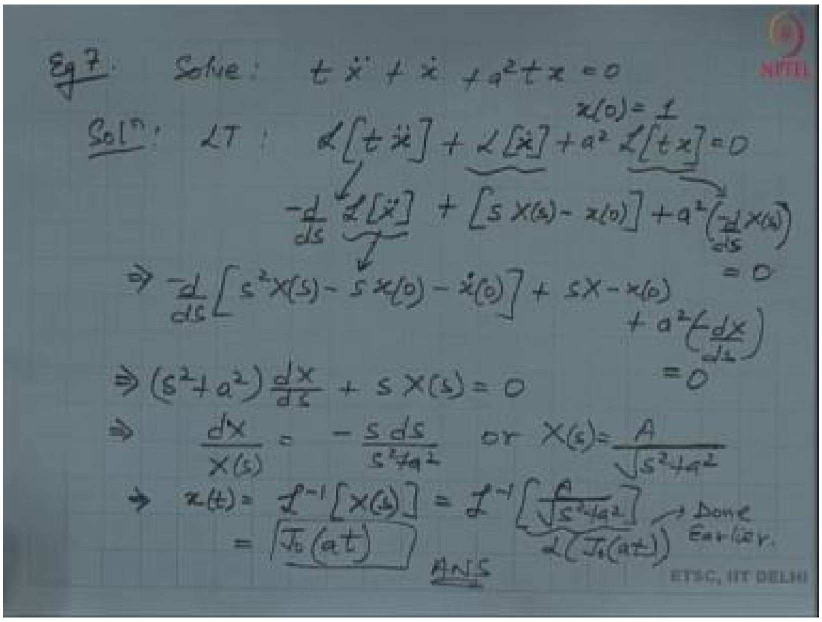
omega sin. So, this is Laplace transform of 1 by omega sin omega t ok. So, which means this function can be finally, if we know this value of this small f I can write this final evaluation of this inverse using my convolution theorem. So, this becomes this final expression is

$$\frac{F}{w}[f * \sin(\omega t)] = L[\frac{1}{\omega}\sin \omega t]$$

Problems for student:

Ho : Danped medium
HO: External Periodic forcing
$(F(t) = \frac{cos\alpha t}{\sin \alpha t}$

So, then well I would like to end the discussion on the simple harmonic oscillator, but there are some other related problems. So, let me just state those problems for students here in this situation I have this harmonic oscillator for the damped medium let us say I ask that we add an extra term due to this damping. So, what happens to the solution the students should also explore that or if we have let us say the harmonic oscillator with some external periodic forcing term right. So, instead of a constant forcing or instead of a forcing we can add the periodic forcing:



Moving on I am going to look at few more cases. So, here is one more example. So, I have to find solve this equation.

Example 7:

Solve:

$$t\ddot{x} + \dot{x} + a^2tx = 0, x(0) = 1$$

Solution:

LT:

$$L[t\ddot{x}] + L[\dot{x}] + a^2L[tx] = 0$$

$$\frac{-d}{ds}\left[s^2x(s) - sx(0) - \dot{x}(0)\right] + [sx(s) - x(0)] + a^2\left(\frac{-d}{ds}x(s)\right) = 0$$

$$\Rightarrow \left(s^2 + a^2\right)\frac{dx}{ds} + sx(s) = 0$$

$$\frac{dx}{x(s)} = -\frac{sds}{s^2+a^2} \text{ or } x(s) = \frac{A}{\sqrt{s^2+a^2}}$$

$$x(t) = L^{-1}[x(s)] = L^{-1}\left\{\frac{A}{\sqrt{s^2+a^2}}\right\}$$

$$= J_0(at)$$

So, I get that x(t) is nothing but the Bessel function of the argument at. So, that is the answer for to this problem.