Integral Transforms and Their Applications
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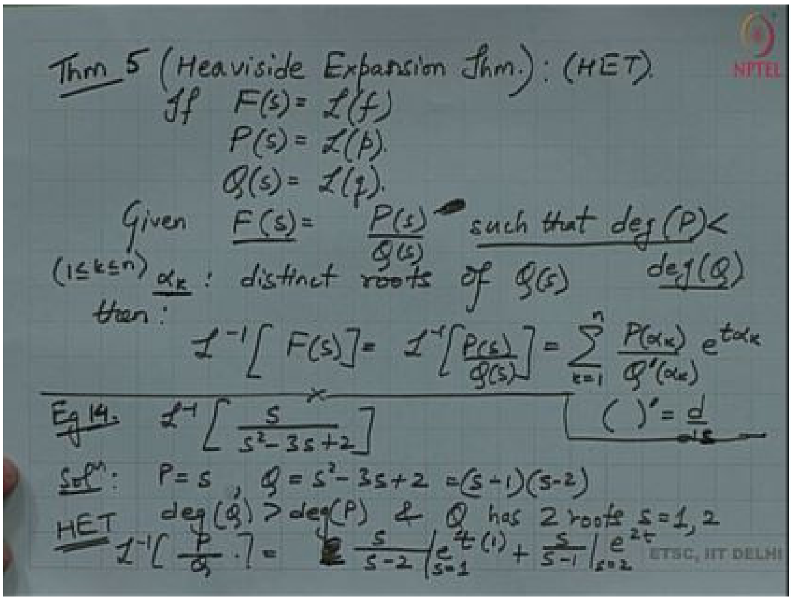
Lecture – 6
Applications of Laplace Transforms- 01

Good morning everyone. So, just to recap, in my last lecture we saw some examples of the Application of Laplace Transform and its properties. And also, specifically how to evaluate the inverse of the Laplace transform. Now, we saw different methods of calculating the inverse, most important among them was the method of Bromwich contour or using the contour integral to evaluate those inverse. Specially, we saw in the last example that the Bromwich contour integral method may not be very easy to apply although, it is quite applicable for any general form of the function. Now, which means that if we do have some nice results or theorems, that can help us to reduce our work to evaluate this Bromwich contours, then it would really simplify are method of evaluating this inverse transforms. So, today in this lecture I am going to talk about some of the nice results or the theorems, which will help us to reduce our work in this contour integral method to evaluate the inverse. And also moving on I am going to talk about some applications of Laplace transform. Specifically using Laplace transform to solve some ODEs and PDEs, thank you.



Theorem 5: Heaviside Expansion Theorem.

$$\text{If } F(s) = \mathcal{L}(f)$$
$$P(s) = \mathcal{L}(p)$$
$$Q(s) = \mathcal{L}(q)$$

Given:

$$F(s) = \frac{P(s)}{Q(s)} \qquad \text{such that } : \deg(P) < \det(Q)$$

$$(1 \leq k \leq n)\alpha_k : \text{ distinct roots of } Q(s)$$

$$\text{then} \quad \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{P(s)}{Q(s)}\right] = \sum_{k=1}^{n} \frac{P(\alpha_k)}{Q'(\alpha_k)} e^{t\alpha_k}$$

$$\text{where : } (\quad)' = \frac{d}{ds}$$

Example 14:

$$\mathcal{L}^{-1}\left[\frac{s}{s^2 - 3s + 2}\right]$$

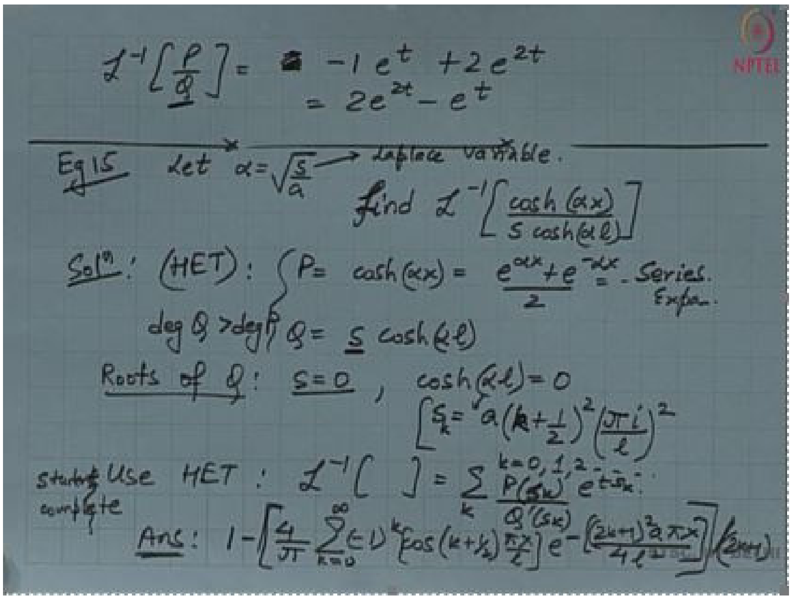
Solution:

$$P = s, Q = s^2 - 3s + 2 = (s-1)(s-2)$$

So, using the Heaviside expansion theorem my inverse is given as follows.

$$\deg(Q) > \deg(P) \quad \text{and} \quad Q \text{ has 2 roots } : s = 1, 2$$

$$\mathcal{L}^{-1}\left[\frac{P}{Q}\right] = \left.\frac{s}{s-2}\right|_{s=1} e^t(1) + \left.\frac{s}{s-1}\right|_{s=2} e^{2t}$$



$$\mathcal{L}^{-1}\left[\frac{P}{Q}\right] = -1e^t + 2e^{2t}$$

$$= 2e^{2t} - e^t$$

Example 15: Let $\alpha = \sqrt{\frac{s}{a}}$ then Find,

$$\mathcal{L}^{-1}\left[\frac{\cosh(\alpha x)}{s\cosh(\alpha \ell)}\right]$$

Solution:By Heaveside Expansion Theorem,

$$P = \cosh(\alpha x) = \frac{e^{\alpha x} + e^{-\alpha x}}{2} \qquad ;\text{series expansion}$$

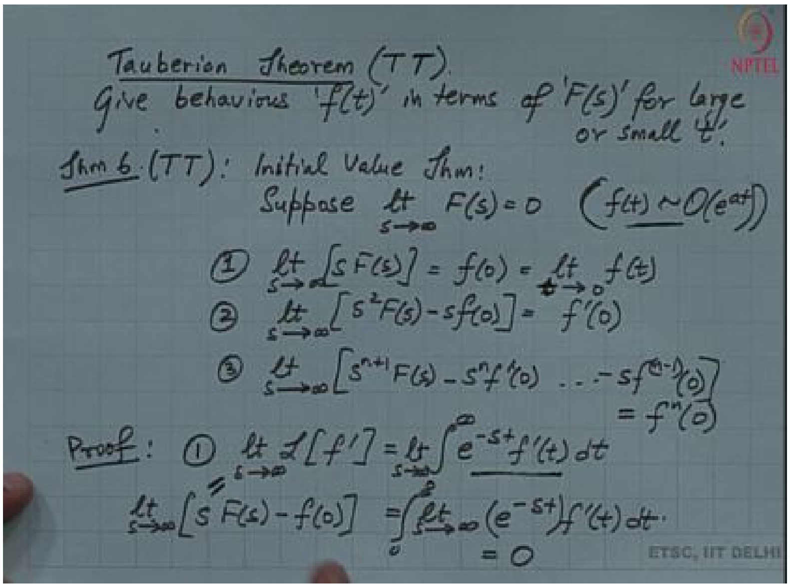
$$Q = s\cosh(\alpha l)$$

Also, $deg(Q) > deg(P)$,
Roots of Q, $s = 0, \cosh(\alpha l) = 0$

$$S_k = a\left(k + \frac{1}{2}\right)^2 \left(\frac{\pi i}{l}\right)^2$$

When I do that I just apply and let me write down the answer after applying this Heaviside expansion theorem, the steps I request the students to complete.

$$1 - \left[\frac{4}{\pi}\sum_{k=0}^{\infty} -1^k [\cos(k + 1/2)\frac{\pi x}{l}\right] e^{-\frac{(2k+1)^2 a\pi x}{4l^2}} /(2k+1)$$



<u>*TuberionTheorem*</u>: So, then many a times, before I move on to my next result. So, I am going to discuss about another very useful result called the Tauberian theorem. So, what does this theorems where is the application of this theorems. So, you see that whenever we evaluate the Laplace transforms, many a times since we know that the Laplace transform needs to for Laplace transform we need to evaluate some infinite integrals many a times we do not have to evaluate or find the value of these integrals at every value of s, the Laplace variable, sometimes all we need is, the value of the Laplace transform at specific points s.

Let us say in the limiting value s going to infinity or in the limiting value s going to 0. So, in that case there is no need to evaluate the entire integral to find those transform and then put the value 0 or limit s tending to infinity. In such a scenario this Tauberian theorems come to a rescue to evaluate this limits.

So, let us see what do these results say. So, the Tauberian theorem as I said they give behaviour, they give the behaviour of your physical function f(t), in terms of your Laplace function F(s), for let us say some asymptotic values of this parameter t, this variable t. Let us say for large or small t.

Theorem 6:Initial Value Theorem,

$$\text{Suppose },\lim_{s\to\infty} F(s) = 0$$

where,

$$f(t) \sim O\left(e^{at}\right)$$

1).

$$\lim_{s\to\infty}[sF(s)] = f(0) = \lim_{t\to 0} f(t)$$

2).

$$\lim_{s\to\infty}\left[s^2F(s) - sF(0)\right] = f'(0)$$

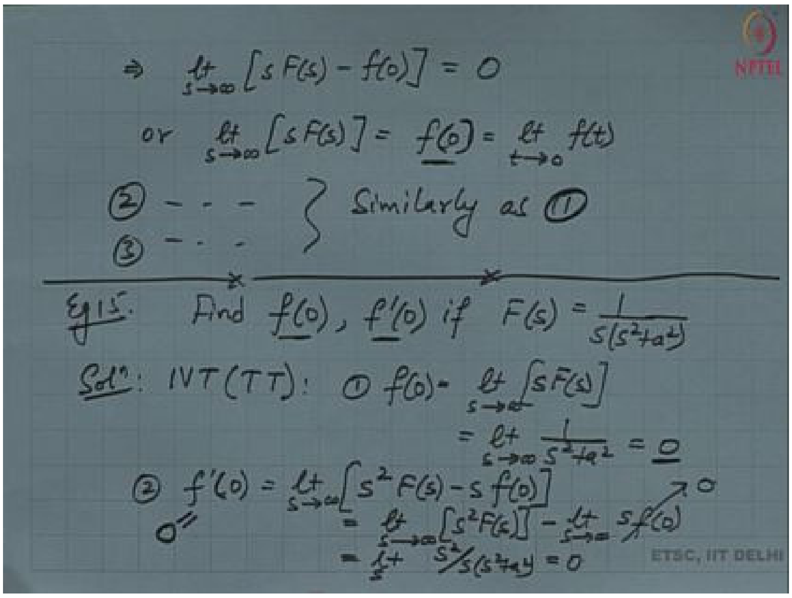
3).

$$\lim_{s\to\infty}\left[s^{n+1}F(s) - s^n f'(0)\ldots - sf^{(n-1)}(0)\right] = f^n(0)$$

Proof 1:

$$\lim_{s\to\infty}\mathcal{L}[f'] = \lim_{s\to\infty}\int_0^\infty e^{-st}f'(t)dt$$

$$\lim_{s\to\infty}\left[s^2F(s) - f(0)\right] = \int_0^\infty \lim_{s\to\infty}\left(e^{-st}\right)f'(t)dt = 0$$



$$\Rightarrow \quad \lim_{s\to\infty}[sF(s) - f(0)] = 0$$

$$\text{or } \lim_{s\to\infty}[sF(s)] = f(0) = \lim_{t\to 0} f(t)$$

So, then the statement for 2 and 3, they follow similarly. So, I leave it to the students to complete the proof of the other 2 statements and follow the same argument as I have used above in the case of the statement 1. So, let us look at a quick application of this initial value theorem, as to where we can apply this right.

Example 15:

$$\text{Find } f(0), f'(0) \text{ if } F(s) = \frac{1}{s\left(s^2+a^2\right)}$$

Solution:The other way of evaluating this and finding these values is to use the Tauberian initial value theorem.

So, using my initial value theorem ,

$$f(0) = \lim_{s\to\infty} [sF(s)]$$

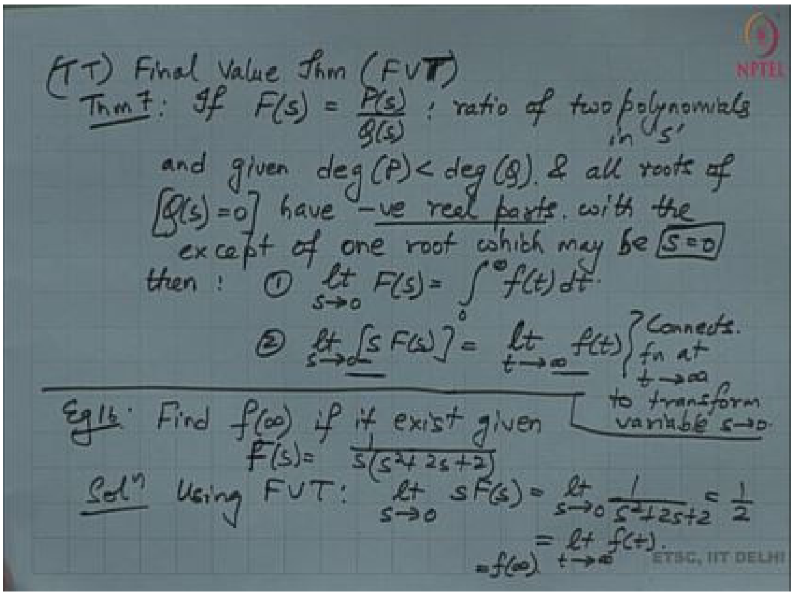
$$= \lim_{s\to\infty} \frac{1}{s^2+a^2} = 0$$

Again using my Tauberian result theorem, this is coming from statement 2 now.

$$f'(0) = \lim_{s\to\infty} \left[s^2F(s) - s^s f(0)\right]$$

$$f'(0) = \lim_{s\to\infty} \left[s^2F(s)\right] - \lim_{s\to\infty} s f(0)$$

$$= \lim_{s\to\infty} \frac{s^2}{s(s^2+a^2)} = 0$$



<u>Final Value Theorem</u>:So, then I have another theorem, the another Tauberian theorem called as the final value theorem. So, let me call theorem 7. So, the Tauberian final value theorem helps us to predict the value at $t = \infty$ using the value for the Laplace variable at $s = 0$.

Theorem 7:

So, what it says is the following suppose I am given that the Laplace transform of F(s) is given by this rational function. So, I can write F(s) as the ratio of two polynomials. So, it is the ratio of 2 polynomials, in the variable s where s is a Laplace variable. And also I am given that the degree of$(P) <$ degree of(Q) and I am also given that all roots of Q, $Q = 0$ have negative real parts. Have negative real parts with the exception of 1 root which may be $s = 0$.

So, what it says is that the denominator is such that all the roots of the denominator Q, have negative real parts, perhaps negative real parts and also perhaps 1 root which may be $s = 0$. So, none of the roots will have positive real parts the most that it can have be that s, one of the root is $s = 0$.

So, in that case my final value theorem says that there are two statements,

1).

$$\lim_{s\to 0} F(s) = \int_0^\infty f(t)dt$$

2).

$$\lim_{s\to 0}[sF(s)] = \lim_{t\to\infty} f(t)$$

Specially, the last statement is quite powerful it helps us to connect the value of this physical function, function for the physical variable in the limiting value to the value of the Laplace transform in this limit. So, this is how the transform and the function are connected from by $s \to 0$ and $t \to \infty$

So, it connects function at $t \to \infty$ in this limit to the transform variable, $s \to 0$. Now, this is specially useful result, because if you were to look at the asymptotic value of some solution all we have to do is to take the Laplace transform of the function. And look at its value $sF(s)$ at near the transformed variable $s = 0$.

So, sometime this limit is easy to evaluate rather than this limit and vice versa. So, let us look at one example quick example here,

Example 16:Find $f(\infty)$ if it exist given,

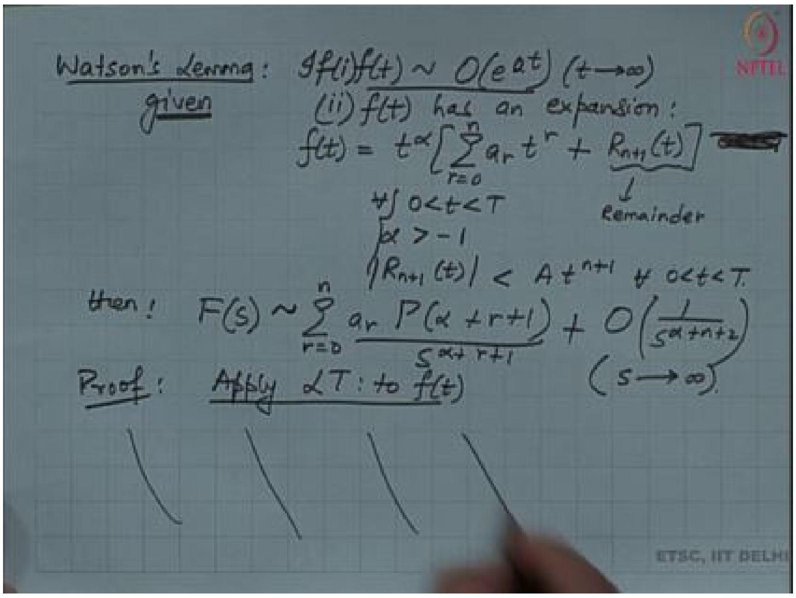
$$F(s) = \frac{1}{s\left(s^2 + 2s + 2\right)}$$

Solution:Using Final Value Theorem,

$$\lim_{s\to 0} s\,\mathrm{F}(s) = \lim_{s\to 0} \frac{1}{s^2 + 2s + 2} = \frac{1}{2}$$

$$= \lim_{t\to\infty} f(t) = f(\infty)$$

So, you see that without evaluating the inverse transform I am able to calculate the value of the function in the limit t tending to infinity using my final value theorem.



So, then I have 1 more result, in this section another useful result called the Watsons lemma. So, what does Watsons lemma says? So, the lemma says that, if I am given that the function f(t) is of exponential order right. So, f(t) is of exponential order in this limit t going to infinity right. And so, let me state all the conditions if I am given this condition right. So, these two are given and the second condition is that f has already has an expansion of the following form.

If,

$$1).f(t) \sim O\left(e^{at}\right) \quad ;(t \to \infty)$$

$$2).f(t) \text{ has an expansion:}$$

$$f(t) = t^{\alpha}\left[\sum_{r=0}^{n} a_r t^r + R_n(t)\right]$$

This is for all t in some interval. So, all t this is for all t in some interval and also that my α has to be bigger than -1.

So, these are all given and also my reminder is such that the absolute value of this reminder is bounded above by some polynomial of degree $n+1$for all T in this interval. So, then the lemma says, the lemma says that the Laplace transform of this function can be approximately written as,

then,

$$F(s) \sim \sum_{r=0}^{n} \frac{a_r \Gamma(\alpha + r + 1)}{s^{\alpha+r+1}} + O\left(\frac{1}{s^{\alpha+n+2}}\right)$$

You see that, so this is in the limit $s \to \infty$.

So, what we see here is that this. So, the proof the proof of this lemma it follows directly by applying my Laplace transform to the function to f(t). We will get the answer right away. So, I am going to leave the proof here leave it to the students to complete the proof.So, moving on.