**Integral Transforms and Their Applications**
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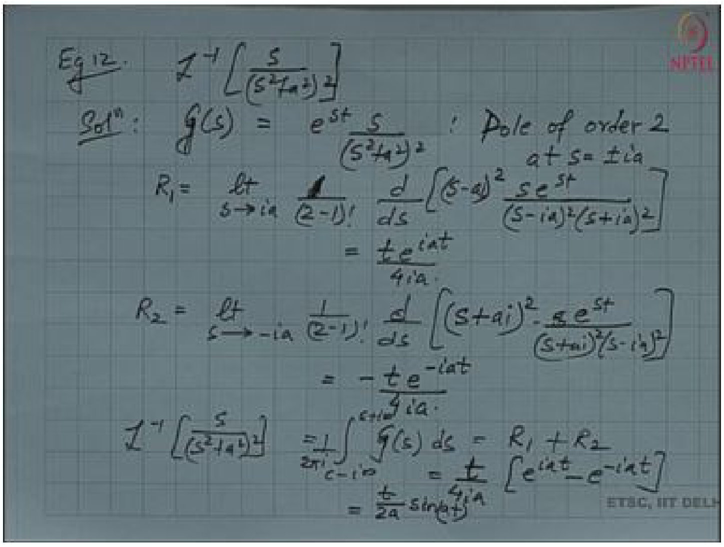
**Lecture – 5**
**Inverse Laplace Transforms, Initial and Final Value Theorems - 03**




Example 12:Find

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]$$

Solution:

$$G(s) = e^{st}\frac{s}{\left(s^2+a^2\right)^2}$$

Notice that, G has a double has a pole of order 2 at s equal to $\pm i$ a. So, these are poles of order 2. So, then if I were to evaluate the residue;

$$R_1 = \lim_{s\to ia}\frac{1}{(2-1)!}\frac{d}{ds}\left[(s-ia)^2\frac{se^{st}}{(s-ia)^2(s+ia)^2}\right]$$

$$= \frac{te^{iat}}{4ia}$$

$$R_1 = \lim_{s\to -ia}\frac{1}{(2-1)!}\frac{d}{ds}\left[(s+ia)^2\frac{se^{st}}{(s-ia)^2(s+ia)^2}\right]$$

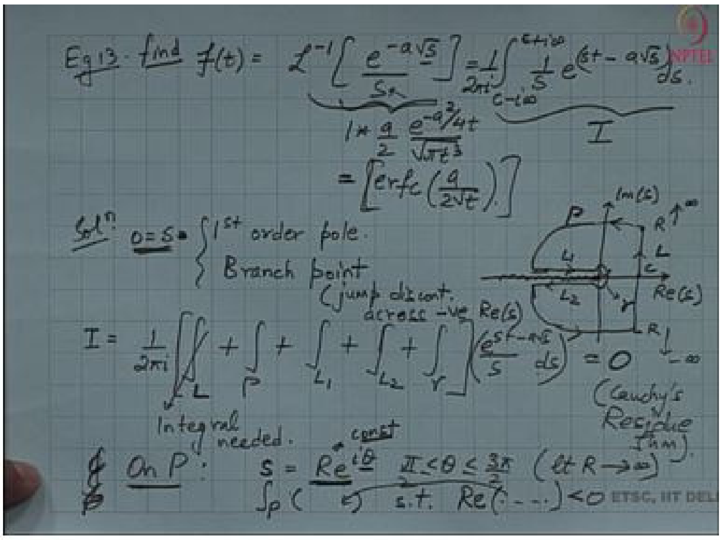
$$= -\frac{te^{iat}}{4ia}$$

$$\mathcal{L}^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(s)ds$$

$$= R_1 + R_2$$

$$= \frac{t}{4ia}\left[e^{iat} - e^{-iat}\right]$$

$$= \frac{t}{2a}\sin(at)$$




Example 13:

$$\text{Find } f(t) = \mathcal{L}^{-1}\left[\frac{e^{-a\sqrt{s}}}{s}\right]$$

$$\frac{1}{2\pi i}\int_{c-t\infty}^{c+i\infty}\frac{1}{s}e^{(st-a\sqrt{s})}ds$$

$$= \left[\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)\right]$$

Solution:Before, I start evaluating this integral. So, all I need to do is that, I need to see that, really my contour integral works well and I need to show you that it indeed gives me this answer ok. So, let us now look at the contour here. So, notice that in this case my contour. So, let me, draw the integral. Let me, draw the curve here, which on which I want to integrate. So, the curve here that, I want to integrate is the following(Refer the slide for explanation).

Let me, draw the curve here, which on which I want to integrate. So, the curve here that, I want to integrate is the following. So, this is the point c and this is, from point R to R where, R goes from -R to R where this, goes to $-\infty$ and this goes to $\infty$ and notice that, in this inverse transform I have a square root function, which is sitting and also that we have $\frac{1}{s}$ function, which is in the. So, s function which is sitting in the denominator. So, the point of 0, so, s equal to 0, is a 1st order pole because of the fact that, it appears in the denominator.

And also, this is also, equal to its also, a branch point. Why is that? Because, notice that, we have a square root function sitting in the numerator and the moment we have a square root function; we know that, there is a sudden jump of the values of s along the negative real axis. So, jump, there is a jump discontinuity across negative real of s. So, hence 0 s equals 0 is a first order pole and s equals 0 is also a branch point. So, if, we were to draw this contour we see that, I have to evaluate the contour, in such a way that, I evaluate I avoid this point 0 and go by bypass it and go around it and also avoid this branch cut and I complete this contour ok.

So, then let me say that, this is these are now there are now, 5 line integrals that, I have to evaluate one is this line integral L. So, my let us say this is my integral I. So, my integral I is,

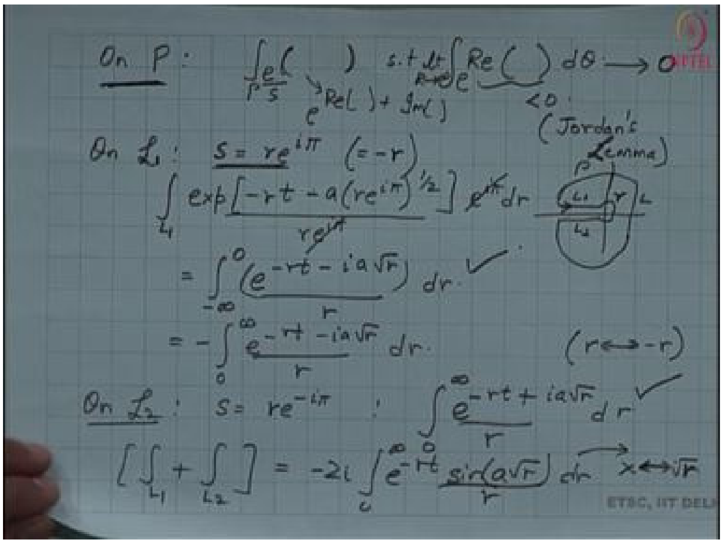
$$I = \frac{1}{2\pi i}\left[\int_L + \int_P + \int_{L_1} + \int_{L_2} + \int_r\right]\left(\frac{e^{st-a\sqrt{s}}}{s}ds\right)$$

Now Cauchy's residue theorem says that since this function is analytic inside this closed curve because there is no singularity as such we have avoided the singularity. So, this integral must give us 0 this is by Cauchy's residue theorem is direct application of that theorem. So, I know that this is the integral you see that this is the integral needed for us to solve.

Now, let us look at this integral the second one. So, let me call. So, the integral over gamma it can be shown that. So, let me say on the curve gamma. So, on the curve gamma this is a curved part of the line integral let me just say that my variable s. So, s is varying from in such a way that it lies on gamma and s the maximum value of s could be R the minimum value of s is minus R.

$$\text{On P :} \quad s = \mathrm{Re}^{i\theta} \quad \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2} (\lim R \to \infty)$$

So, we can show that in this scenario my integral over gamma of if I just plug this value of s and then of course, evaluate this infinitesimal element d s it is R is this value R is a constant eventually we take a limit of this constant limit R tending to infinity it is this variable which is changing with as s. So, that the derivative of s is with respect to the derivative of theta. So, when we substitute this value in this integral I am going to get something inside this integral such that what I see is that the real part of this thing inside this integral the real part is negative. So, then as we take. So, let me just repeat what I just said.




So, on the curve gamma my integral is such that the real part of this integral real part. So, I can break it down into the real part and the imaginary part. The real part is such that this real part is negative and when we take the limit R tending to infinity. So, in the limiting case this vanishes this real part vanishes and in we can see that well to begin with we also had an exponential. So, we have an exponential function times some argument divided by some s and when we plug in our substitution we can see that it becomes the following it becomes e to the power real of something plus e to the power imaginary of something. It can be shown that the real part of that exponential is negative and when we take r tending to infinity then this goes to 0. So, in short the integral over this curve gamma can be shown to be 0 and again this is via the direct application of Jordan's lemma where we see that over the curved part this is 0. Now let us look at the integral over the line L 1 ok. So, again I am drawing this curve again. So, I have the following. So, I am just exaggerating my contour. So, I have line L 1 this is line

L 2 this is gamma this is line L and this is small gamma. So, on L 1 let me choose my variable s.

So,this time we are traversing from negative infinity to 0. So,

$$\text{On L1 :} \quad s = re^{i\pi} \quad (= -r)$$

$$\int_{L1} \exp \left[ \frac{\left[-rt - a\left(re^{i\pi}\right)^{1/2}\right]}{re^{i\pi}} \right] e^{i\pi} dr$$

$$= \int_{-\infty}^{0} \frac{\left(e^{-rt - ia\sqrt{r}}\right)}{r} dr$$

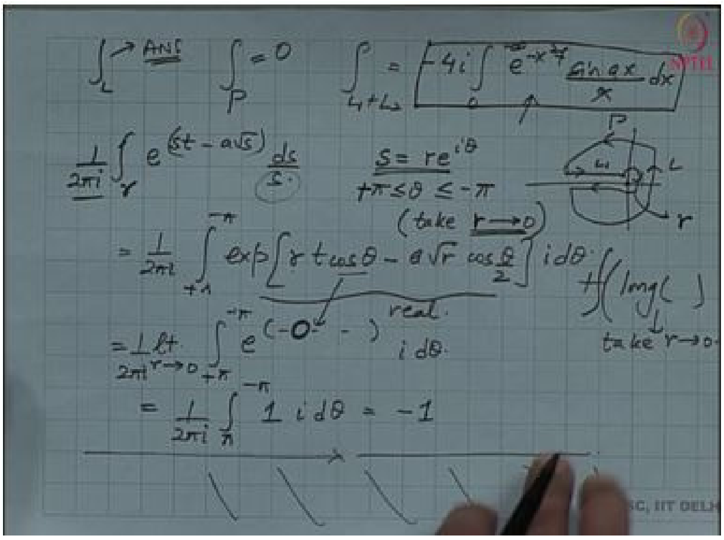
$$= -\int_{0}^{\infty} \frac{e^{-rt} - ia\sqrt{r}}{r} dr$$

$$\text{On L2 :} \quad s = re^{-i\pi} : \int_{0}^{\infty} \frac{e^{-rt} + ia\sqrt{r}}{r} dr$$

$$\left[\int_{L1} + \int_{L2}\right] = -2i \int_{c}^{\infty} \frac{\sin(a\sqrt{r})}{r} dr$$

So, then I can do a change of variables let me do let me change x replace $x = \sqrt{r}$. So, I am instead of $\sqrt{r}$ I am going to replace it with x. So, I get that this integral over L 1 plus integral over L 2 becomes:

$$-4i \int_{0}^{\infty} e^{-x^2 t} \frac{\sin ax}{x} dx$$




And then finally, so, finally, what I get is now let us now. So, I have evaluated over L. So, that is my answer that I want and I have evaluated the integral over gamma capital gamma and I have shown that this is equal to 0, I have evaluated the integral over L 1 plus L 2 and I have shown that

And then I have to see what is the integral over this small gamma the small curve. So, again let me draw this curve quickly. So, I have this thing right. So, I am talking about. So, this is my curve this is over L, this is over gamma and this is the curve I am talking about is small gamma. So, integral over gamma that I get is

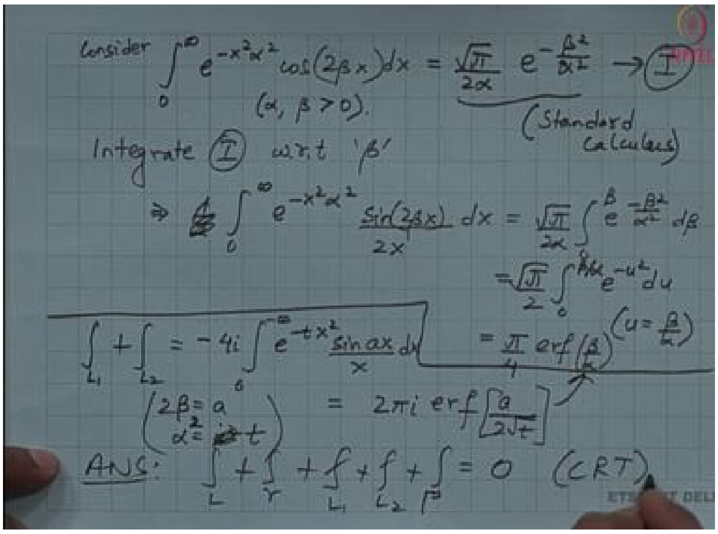
$$\frac{1}{2\pi i}\int_{\gamma} e^{(st-a\sqrt{s})}\frac{ds}{s} \quad , \quad \frac{s = re^{i\theta}}{+\pi \leqslant \theta \leqslant -\pi}$$

$$= \tfrac{1}{2\pi i}\int_{+\pi}^{-\pi} \exp\left[rtcos\theta - a\sqrt{r}\cos\tfrac{\theta}{2}\right] id\theta + \int Im(.)$$
$$= 1\ell + \int_{+\pi}^{-\pi} e^{(-0)-1} id\theta$$

$$= \frac{1}{2\pi i}\lim_{r\to 0}\int_{+\pi}^{-\pi} e^{(-0.....)} - id\theta$$

$$= \frac{1}{2\pi i}\int_{\pi}^{-\pi} 1 id\theta = -1$$

So, finally, what I have is I need to post process this result a little bit more over L 1 and L 2. So, notice that I have this. So, let me just digress; let me just digress a little bit because I want to evaluate this integral as well. So, I am slightly digressing from my question at hand.




And I want to see what is the value of this integral from 0 to infinity, notice that this integral is our standard integral which is over the real axis. So, I am going to consider this integral. So, I know that the value is I let me choose alpha and beta to be positive and I know that this is also equal to

$$\int_0^{\infty} e^{-x^2\alpha^2\cos(2\beta x)dx} = \frac{\sqrt{\pi}}{2\alpha}e^{-\frac{\beta^2}{\alpha^2}} \to \text{(I)}$$

So, this is coming from standard calculus. So, students who want to see where we are getting this answer from should try to either derive it if they can substitute alpha times x to be y and then use integration by parts taking one of the functions as first the other one as the second function to arrive at this result ok. So, then let me call this as my result I. So, I am going to integrate I with respect to the variable $\beta$.

$$\int_0^\infty e^{-x^2\alpha^2} \frac{\sin(2\beta x)}{2x} dx = \frac{\sqrt{\pi}}{2\alpha} \int_0^\beta e^{\frac{-\beta^2}{\alpha^2}} d\beta$$

$$= \frac{\sqrt{\pi}}{2} \int_0^{\beta/\alpha} e^{-u^2} du$$

$$= \frac{\pi}{4} \operatorname{erf}\left(\frac{\beta}{k}\right) \qquad where, u = \frac{\beta}{\alpha}$$

So, then coming back notice that my integral over L 1. So, I am coming back to my problem my integral over L 1 and L 2

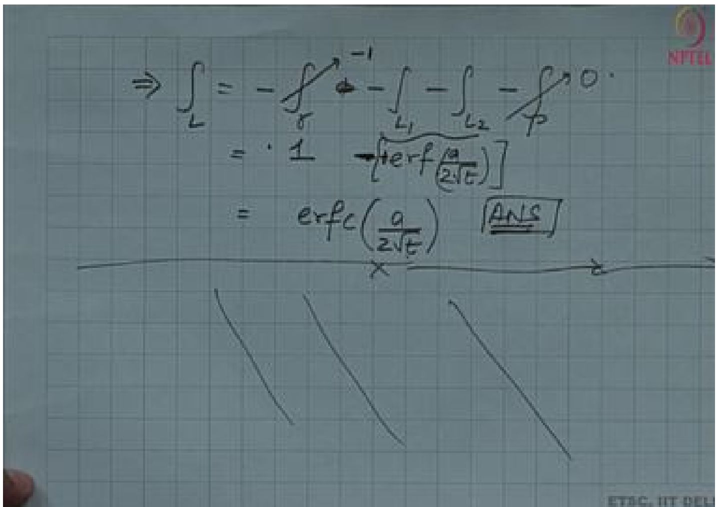
$$\int_{L_1} + \int_{L_2} = -4i \int e^{-tx^2} \frac{\sin ax}{x} dx$$

$$= 2\pi i \operatorname{erf}\left[\frac{a}{2\sqrt{t}}\right]$$

Ans:

$$\int_L + \int_r + \int_{L1} + \int_{L2} + \int_\Gamma = 0$$

So, finally, the all the when I have to match all the integrals. So, the answer that I have get is the integral over L plus the integral over gamma plus the integral over L 1 plus the integral over L 2 plus the integral over capital gamma all these integrals I am going to get one of them was. So, this one plus this one I get that this is equal to well I have this is equal to 0. So, what I get is using Cauchy's residue theorem right.




$$\int_L = -\int_r - \int_{L_1} - \int_{L_2} - \int_\Gamma$$

$$= 1.erf\left(\frac{a}{2\sqrt{t}}\right)]$$

So, let me just end this lecture at this point in the next lecture i am going to provide you with some useful results known as the Tauberian theorems or the Watsons lemma which helps us to calculate some of these inverse transforms in a very neat or a compact fashion. Thank you very much.