Integral Transforms and Their Applications

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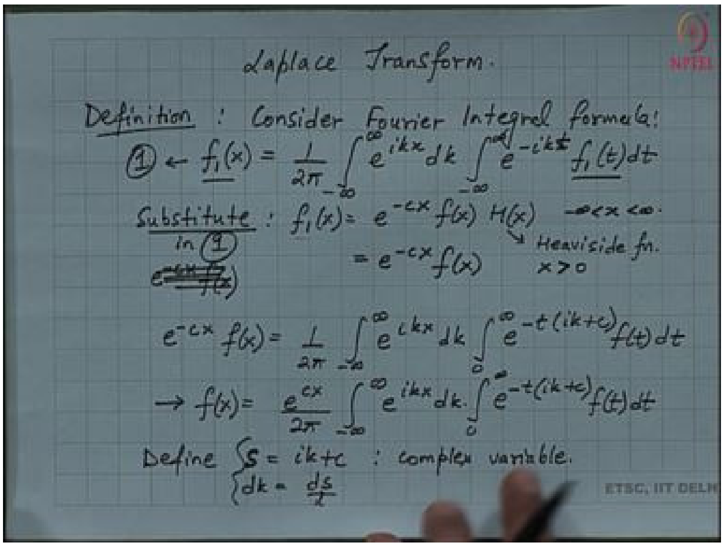
Lecture – 4
Applications of Laplace Transforms Part - 02

So, in my next set of notes and next set of you know lectures I am going to talk about Laplace transform and how to evaluate Laplace transforms. So, moving ahead.



So, let me just start with the basic definition of Laplace transform. So, since we do not know the definition of a priory I am going to start the definition of Laplace transform by considering the Fourier integral formula. So, I am going to derive my Laplace transform by the Fourier integral formula. So, consider the Fourier integral formula of this function $f_1(x)$.So, this is given by:

$$f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{-\infty}^{-\infty} e^{-ikt} f_1(t) dt \tag{1}$$

So, that is my Fourier integral formula.

Now, let us substitute. So, let us look at a particular situation where I substitute $f_1(x)$ to be equal to let us say,

$$f_1(x) = e^{-cx} f(x) H(x) \quad -\infty < x < \infty$$

So, when I do that I substitute here and I substitute here, I get the following expression. So, well I can always write this function $f_1(x)$ in terms of Heaviside function this is for all x or this is also equal to from this simple expression

$$= e^{-cx} f(x)$$

Well, if I do not want to use this Heaviside function expression. So, then I substitute this function, substitute, let us say this is my expression 1.

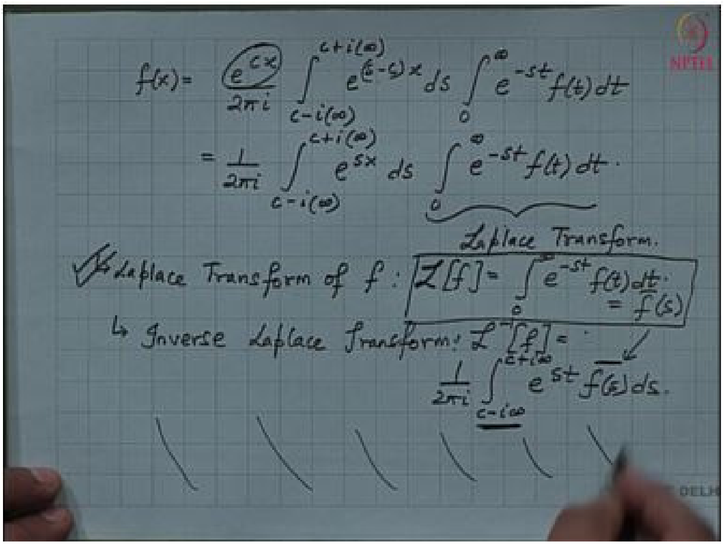
So, substitute in 1, I get the following expression;

$$e^{-cx} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk \int_{0}^{\infty} e^{-t(ik+c)} f(t) dt$$

$$f(x) = \frac{e^{cx}}{2\pi} \int_{-\infty}^{0} e^{ikx} dk \cdot \int_{0}^{\infty} e^{-t(ik+c)} f(t) dt$$

$$\text{Define} \begin{cases} S = ik + c \\ dk = \frac{ds}{i} \end{cases} : \text{ complex variable}$$

So, I am going to substitute all these expressions in this particular expression for f(x).



So, I get the following new expression.

$$f(x) = \frac{e^{cx}}{2\pi i} \int_{c-i(\infty)}^{c+i(\infty)} e^{(s-c)x} ds \int_{0}^{\infty} e^{-st} f(t) dt$$

Now notice that in the new variable in the old variable k was from negative infinity to infinity, in the new variable my s is from $c - i(\infty)$ to $c + i(\infty)$. Now, student should not confuse with this fact that what how is this possible that we are adding finite number to an infinite number. Well notice that c is a real constant.

So, I am going to talk about this sort of an integration where the real constant is finite while the imaginary value goes to infinity. So, this sort of an integral will be dealt slightly later in another lecture. So, then starting continuing to substitute I get,

$$= \frac{1}{2\pi i} \int_{c-i(\infty)}^{c+i(\infty)} e^{sx} ds \int_{0}^{\infty} e^{-st} f(t) dt$$

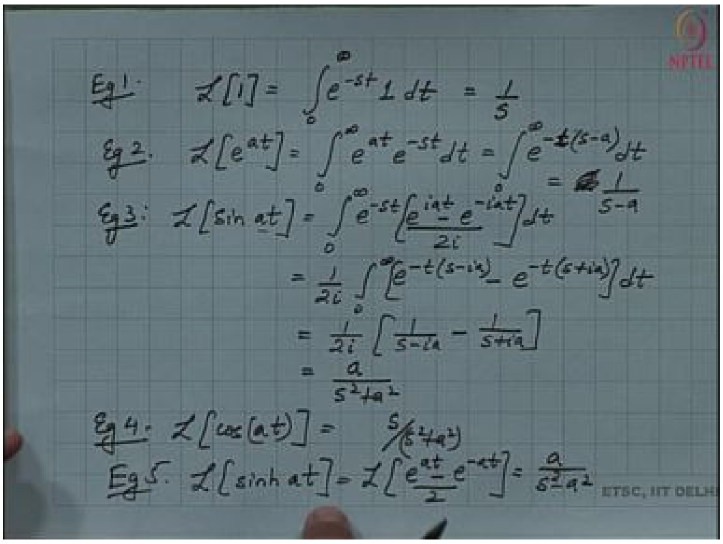
Now, notice that I am going to define my Laplace transform of f; let us say Laplace transform. So, the Laplace transform is denoted by this integral ok. So, in particular the Laplace transform:

$$L[f] = \int_0^\infty e^{-st} f(t) dt = \overline{f}(s)$$

Similarly, the inverse transform the inverse Laplace transform also denoted by L inverse of f is denoted by:

$$L^{-1}[f] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \overline{f}(s) ds$$

So, I am going to denote this Laplace transform also by $\overline{f}(s)$. So, that is this $\overline{f}(s)$. So, now, again briefly this integral limits are not infinite as such and I am going to talk about this inverse transform in another lecture. So, for the time being I am going to focus on Laplace transform of functions. So, I have introduced the Laplace transform and the definition of Laplace transform in this expression here. So, let us continue describing and looking at some common Laplace transform of some common functions here; ok.



So, to begin with here are some simple examples:

Example 1:

$$L[1] = \int_0^\infty e^{-st} 1 \mathrm{dt} = \frac{1}{s}$$

Example 2:

$$L\left[e^{at}\right] = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{-t(s-a)} dt = \frac{1}{s-a}$$

Example 3:

So, then few more examples we can continue like this. We have the Laplace transform of let us say the sine function. So, sinat. So, I can always introduce instead of sin, I can always use the fact that my sine function is in terms of the exponential functions. So, the Euler's formula;

$$L[\sin at] = \int_0^\infty e^{-st} \left\{ \frac{e^{iat} - e^{-iat}}{2i} \right] dt$$

$$= \frac{1}{2i} \int_0^\infty \left[e^{-t(s-ix)} - e^{-t(s+ix)} \right] dt$$

$$= \frac{1}{2i} \left[\frac{1}{s-ia} - \frac{1}{s+ia} \right]$$

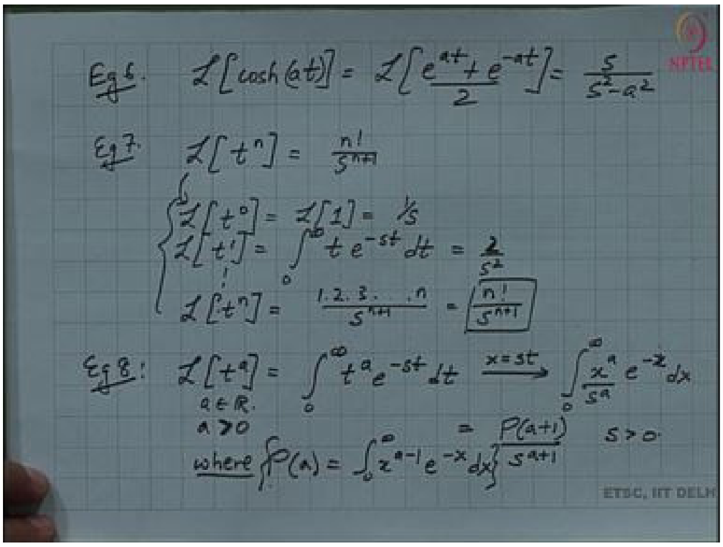
$$= \frac{a}{s^2 + a^2}$$

Example 4:Similarly, the Laplace transform of the cosine of the cosine function is given by,

$$L[\cos(at)] = \frac{s}{(s^2 + a^2)}$$

Example 5:

$$L[\sinh at] = \left[\frac{e^{at} - e^{-at}}{2}\right] = \frac{a}{s^2 - a^2}$$



Then, I have few more examples,

Example 6:

$$L[\cosh(\alpha t)] = \left[\frac{e^{at} + e^{-at}}{2}\right] = \frac{s}{s^2 - a^2}$$

Example 7:

$$L\left[t^n\right] = \frac{n!}{s^{n+1}}$$

$$\begin{array}{l} L\left[t^0\right] = L[1] = \frac{1}{s} \\ L\left[t'\right] = \int_0^\infty te^{-st}dt = \frac{2}{s^2} \\ \mathcal{L}\left[t^n\right] = \frac{1.2.3\cdots n}{s^{n+1}} = \left[\frac{n!}{s^{n+1}}\right] \end{array}$$

Example 8:

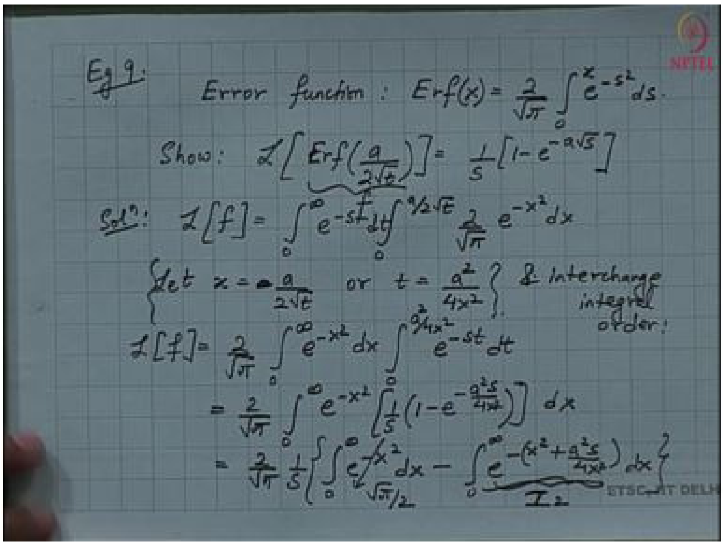
$$\mathcal{L}\left[t^a\right] = \int_0^\infty t^a e^{-st} dt \xrightarrow{x = st} \int_0^x \frac{x^a}{s^a} e^{-x} dx$$

$$= \frac{\Gamma(a+1)}{s^{a+1}} \quad s > 0$$

So, now, what is gamma? The gamma function of any number a is given by this integral. So, where, gamma function is given by,

$$\text{where } \Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx \Big\}$$

So, continuing let us look at slightly more complicated situation where we have to calculate the Laplace transform of a function which is not very trivial.



So, let me introduce the function known as the error function. So, what is error function? So, let me call it as a denoted by

Example 9:

$$Erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}ds$$

$$\text{Show:} \quad 2\left[\text{Erf}\left(\frac{a}{2\sqrt{t}}\right)\right] = \frac{1}{s}\left[1 - e^{-a\sqrt{s}}\right]$$

Solution:

$$L[f] = \int_0^\infty e^{-st}dt \int_0^{a/2\sqrt{t}} \frac{2}{\sqrt{\pi}}e^{-x^2}dx$$

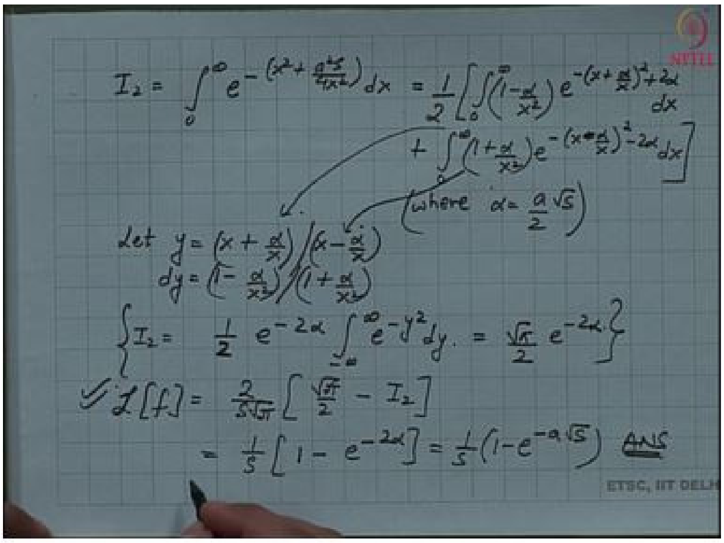
$$\text{Let } x = -\frac{a}{2\sqrt{t}} \text{ or } t = \frac{a^2}{4x^2}\Big\} \begin{array}{c}\text{\& interchange}\\ \text{integrel order}\end{array}$$

$$L[f] = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}dx \int_0^{a^2/4x^2} e^{-st}dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-x^2}\left[\frac{1}{s}\left(1 - e^{-\frac{a^2 s}{4x^2}}\right)\right]dx$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} \left[\int_0^\infty e^{-x^2} dx - \int e^{-\left(x^2 + \frac{a^2 s}{4x^2}\right)} dx \right]$$

this one I can immediately find that the value of this is $\frac{\sqrt{\pi}}{2}$ andI have to evaluate this other integral, let us call this other integral as I_2. So that is what I want to evaluate



So, I have the expression for I_2 as follows:

$$I_2 = \int_0^\infty e^{-\left(x^2 + \frac{a^2 s}{4x^2}\right)} dx = \frac{1}{2} \left[\int_0^\infty \left(1 - \frac{\alpha}{x^2}\right) e^{-(x + \frac{\alpha}{x})^2 + 2\alpha} dx + \int_0^\infty \left(\frac{1 + \alpha}{x^2}\right) e^{-\left(x - \frac{\alpha}{x}\right) - 2\alpha} dx \right]$$

$$\left(\text{where } \alpha = \frac{a}{2}\sqrt{s} \right)$$

$$\text{Let } y = \left(x + \frac{\alpha}{x}\right) / \left(x - \frac{\alpha}{x}\right)$$

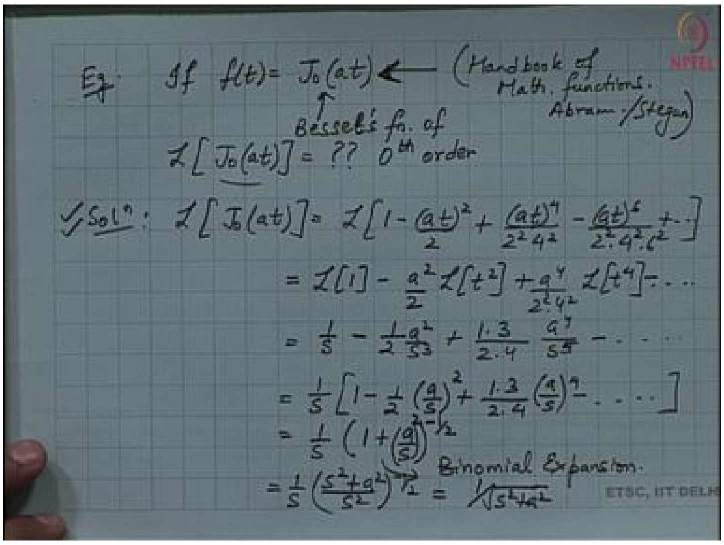
$$dy = \left(1 - \frac{\alpha}{x^2}\right) / \left(1 + \frac{\alpha}{x^2}\right)$$

$$\left\{ I_2 = \frac{1}{2} e^{-2\alpha} \int_{-\infty}^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2} e^{-2\alpha} \right\}$$

$$\mathcal{L}[f] = \frac{2}{s\sqrt{\pi})} \left[\frac{\sqrt{\pi}}{2} - I_2 \right]$$

$$= \frac{1}{s} \left[1 - e^{-2\alpha}\right] = \frac{1}{s} \left(1 - e^{-\alpha\sqrt{s}}\right)$$

So, this was this example was particularly taken because we wanted to show how to use Laplace transforms, specifically, how to use the different types of integrals and typically you will see that, in evaluating this Fourier transforms or Laplace transforms you have to resort to integration techniques let say integration by parts, partial fractions and using some well known integrals. So, it is necessary that, most basic knowledge of calculus is known to the students specially how to integrate certain functions. So, moving on then, let me just show another example.



Example: So, if my function f(t) is given by this $J_0(at)$. So, what is J_0?J_0 is the so called Bessel's function of zeroth order. So, Bessel's function of zeroth order; ok. So, this is. So, I have to find what is my Laplace transform of the Bessel's function, so, this is something I have to find. So, again students are requested to look at the handbook, the handbook of mathematical functions; that is what I have listed in my handout in my course description page. So, this is by Abramowitz and Stegan. So, please, refer to this book to see the exact expression for this Bessel's function of order 0.

Now, I will highlight I will show you the expression for the Bessel's function in the series form. So, to begin with my Laplace transform of this Bessel's function is the following expression.

Solution:

$$\mathcal{L}\left[J_0(at)\right] = \mathcal{L}\left[1 - \frac{(at)^2}{2} + \frac{(at)^4}{2^2.4^2} - \frac{(at)^6}{2^2 \cdot 4^2 \cdot 6^2} + ...\right]$$

$$= \mathcal{L}[1] - \frac{a^2}{2}\mathcal{L}\left[t^2\right] + \frac{a^4}{2^2 \cdot 4^2}\mathcal{L}\left[t^4\right] - .....$$

$$= \frac{1}{s} - \frac{1}{2}\frac{a^2}{s^3} + \frac{1.3}{2.4}\frac{a^4}{s^5} - \dots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{a}{s}\right)^2 + \frac{1.3}{2.4}\left(\frac{a}{s}\right)^4 - \dots\right]$$

$$= \frac{1}{s}\left(1 + \left(\frac{a}{s}\right)^2\right)^{-1/2}$$

Notice that, this is nothing but the binomial expansion.So, if I use a binomial expansion of this factor, we, I am going to get this series solution this series expression above this.

$$= \frac{1}{s}\left(\frac{s^2+a^2}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{s^2+a^2}}$$

So, that is my Laplace transform of the Bessel's function of the zeroth order.