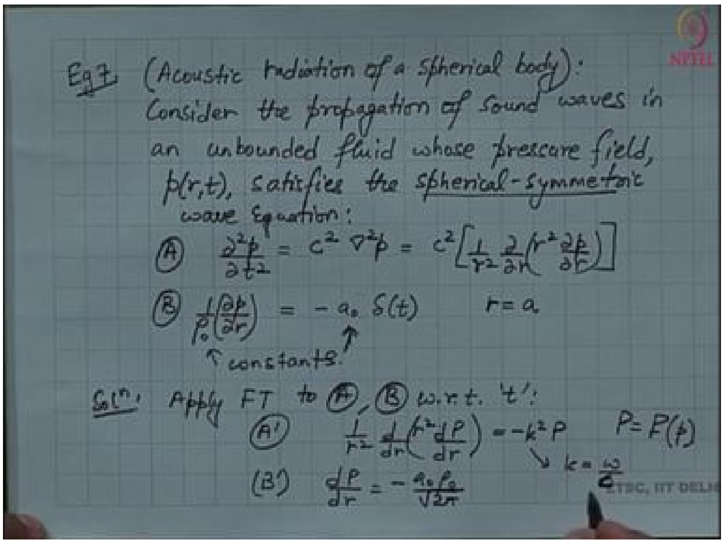
Integral Transforms and Their Applications
Prof. Sarthok Sircar
Department of Mathematics
Indraprastha Institute for Information Technology, Delhi

Lecture – 4
Applications of Laplace Transforms Part - 01

So, good afternoon everyone; so, in today's lecture I am going to continue to discuss some more applications of Fourier transforms. Specially, applications related to Acoustic radiations to water waves that, we had discussed in the last lecture as well and moving on I am going to introduce the theory of Laplace transform and also look at some examples in Laplace transform.



So, starting let us continue our examples that we were discussing. So, I am going to continue the sequence of examples that we were discussing in the last lecture. So, in this next example I am going to talk about the case of Acoustic radiation, Acoustic radiation of a spherical body. So, I am going to solve for the sound waves. So, Acoustic means sound. So, sound waves that are generated by a spherical body, which are immersed in a fluid. So, let us say the problem is we have two. So, again let me write down the statement.

So, consider the propagation of sound waves; So, consider the propagation of sound waves in an unbounded fluid whose pressure, whose pressure field let us say we denote the pressure field by p(r,t) satisfies the spherically symmetric wave equations. So, spherically symmetric wave equation ok so, what is that equation, the equation is given by:

$$\text{(A)}\ \frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi = c^2\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right)\right]$$

So, notice that, there is no θ or ϕ dependence of the derivative on this expression. So, the expression is completely dependent on r and t ok. Then, let me just denote this expression as:

$$\text{(B)}\ \frac{1}{\rho_0}\left(\frac{\partial p}{\partial r}\right) = -a_0\delta(t), \quad r = a$$

I must specify my boundary conditions and initial conditions. So, I have that one by. So, the bound one of the boundary condition is as follows. So, this is as you can see I am using a derivative of the pressure. This will be a Neumann boundary condition. So, this is given to be equal to $-a_0\delta(t)$.So, this is at r radius r equal to a. So, these are constants. So, these are all constants. So, r at r equal to a I can, I have specified my gradient of pressure.

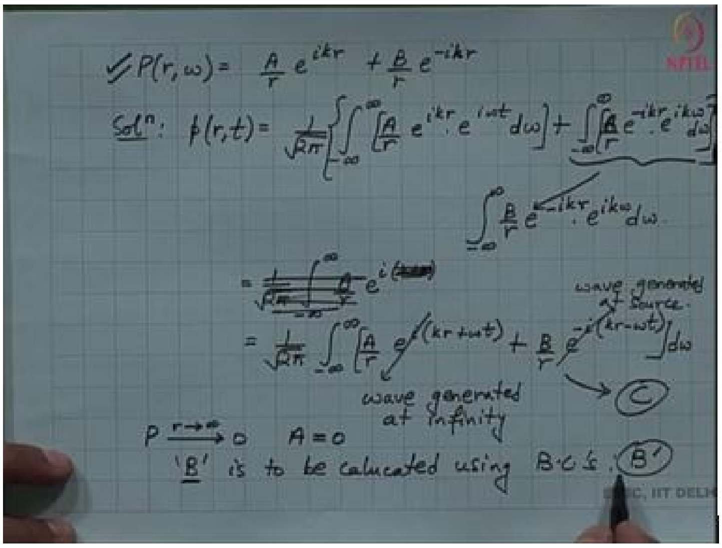
So, then let us start. Well, of course, you know we have. Well, with this with these two expressions we can start finding the solution to this PDE. So, again we apply Fourier transform, we apply Fourier transform to both A and B and I get the following new set of equations.

$$(\mathrm{B}') \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dP}{dr}\right) = -k^2P \quad F = F(p) \qquad \text{where,} \quad k = \frac{w}{c}$$

So, we have applied the Fourier transform to A and B; we have applied it with respect to the variable t this time and then my expression B prime, it reduces to:

$$(B') \quad \frac{dP}{dr} = -\frac{a_0 \cdot P_0}{\sqrt{2\pi}}$$

So, notice that, now, the PDE has become an ODE; So, I am, since I am transforming with respect to the frequency. In the frequency domain my, my transformed variable is omega; so, then, in that case.



So, solving for these two, I get the following expression:

$$P(r, w) = \frac{A}{r}e^{ikr} + \frac{B}{r}e^{-ikr}$$

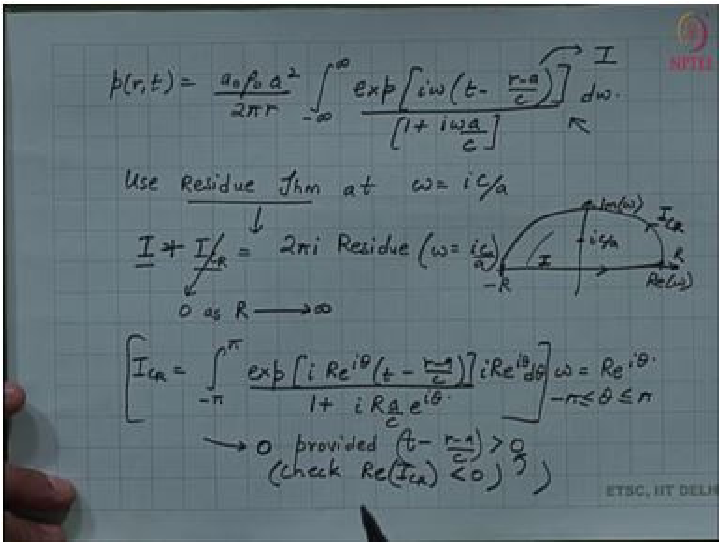
Solution: finally, my solution the pressure wave is given by:

$$p(r, t) = \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{A}{r}e^{ikr}e^{i\omega t}d\omega\right] + \int_{-\infty}^{\infty}\left[\frac{B}{r}e^{-ikr}e^{ik\omega}d\omega\right]\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{A}{r}e^{i(kr+\omega t)} + \frac{B}{r}e^{-i(kr-\omega t)}\right]d\omega \qquad (C)$$

So, notice that, there are two expressions in this integrand. Specifically, this first expression is due to the wave, which is generated at infinity and the second expression is the wave is due to the wave generated at the source. So, what do I mean by that is that, suppose you have a disturbance in as r goes to infinity? Then, you are going to retain this first term and suppose you are, you have some disturbance at r equal to 0, then you are going to retain the second term. So, specifically, since I assume. Well, if you recall, I have not this is a second order PDE and I have to provide two conditions to describe, these two constants.

So, the second condition is that, P goes to 0 as r goes to ∞ and in that case, I have to take my A to be equal to 0. Now, if I do that my B, my coefficient B is to be calculated using my boundary conditions that I have described. So, that was specifically B prime that, I have written in the transformed plane. So, after I do all those calculations. So, let us say that this expression is my expression C. So, and then, I substitute B here from the boundary condition that, I had already calculated in this expression C.



I get the solution as follows, I get my pressure wave:

$$p(r,t) = \frac{a_0 p_0 a^2}{2\pi r} \int_{-\infty}^{\infty} \frac{\exp\left[iw\left(t - \frac{r-a}{c}\right)\right]}{\left(1 + i\frac{wa}{c}\right]} dw$$

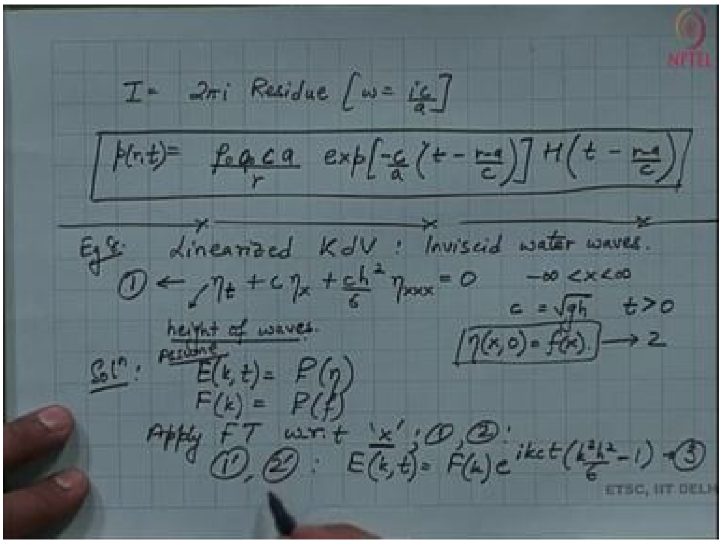
So, now, to further evaluate this integral, we see that again, the idea is to use the residue theorem right. So, we need to use residue theorem at omega equal to, look at the denominator the denominator vanishes at omega equal to this value, which is $\frac{ic}{a}$ right. So, I have. Let us say that this integral is I. so, my I is equal to, let me draw the diagram. So, if I have this diagram, where I am talking about, let us say real omega versus imaginary omega. (Refer the respective slide for better understanding). So, my ω the point of singularity sitting somewhere here and now if I am plotting, if I am trying to figure out the solution forty positive, I must have a contour in the positive half of the plane right. So, let me say this is, since this is my integral i and let us say that this is my integral over CR right; so, my integral I plus my integral. Well, let me say that this is Use Residue theorem at $\omega = \frac{ic}{a}$:

$$I + I_{c_R} = 2\pi i \text{ Residue} \quad (\omega = ic)$$

It further you know, we have to further show the fact that this I_{c_R} goes to 0 as so, of course, this is a circle from minus R to R, the span being from minus R to R. So, as R goes to infinity, this I_{c_R} goes to 0. So, let us look at this, this integral quickly; so, I_{c_R} would be the following integral, integral from. Well, so, I have the in integral expression here; notice this, integral expressions.

$$I_{c_R} = \int_{-\pi}^{\pi} \frac{\exp\left[iRe^{i\theta}\left(t - \frac{r-a}{c}\right)\right] iRe^{i\theta} d\theta}{1 + iR\frac{a}{c}e^{i\theta}} \quad \text{for,} \begin{array}{c} \omega = Re^{i\theta} \\ -\pi \leqslant \theta \leqslant \pi \end{array}$$

So, all we have to make sure is. Well, this integral is from minus π to π; so, this is the theta integral. We can see that, this integral will go to 0 provided, provided I have the factor $t - \frac{r-a}{c}$, this factor is positive right. Particularly, you can check, check that the real part, the real part of this I_{c_R} is negative in this situation, when we have this inequality being satisfied, my real part is negative and when R goes to ∞.



So, then, my integral I is equal to:

$$I = 2\pi i \quad \text{Residue} \left[w = \frac{ic}{a}\right]$$

After substituting I get that my solution, which is the pressure wave given by.So, I am just substituting the residue and using the residue theorem and simplifying the expression in the integral:

$$H(n+) = \rho_0\, a_0 \text{exp}\left[-\frac{c}{a}\left(t - \frac{r-a}{c}\right)\right] H\left(t - \frac{r-a}{c}\right)$$

So, that is my solution to the Acoustic radiation problem.

So, this is fully integrable because we were able to find the residue to that single integral. Moving on, we have few more another example. This time I am going to introduce the example of linearized KdV equation. Now, KdV stands for the full form Korteweg-de Vries equation and this is constant, very commonly used for solving problems related to water waves. So, in this equation, I am going to talk about KdV equations for the linearized scenario or when the coefficients are held constant.

Example 8: So, I have the following linearized; so, this is for particularly for non viscous or inviscid water waves ok. So, my KdV equation looks as, looks like following:

$$\eta_t + c\eta_x + \frac{ch^2}{6}\eta_{xxx} = 0 \qquad (1) \quad \text{for,} \begin{array}{c} -\infty < x < \infty \\ c = \sqrt{gh} \quad t > 0 \end{array}$$

further I have a bound, I have an initial condition

$$\eta(x,0) = f(x) \tag{2}$$

notice that η in the physical term η denotes the height of the water waves.

So, this is the height, the height of the water waves and then, let me start the solution by saying that the Fourier transform of, of your solution is denoted by E; so, the Fourier transform of η, we denoted by E and the Fourier transform of my initial condition is denoted by capital f. So, then what I get is, when I apply my Fourier transform, apply Fourier transform with respect to, with respect to x, the variable x this time. I get my solution as follows:

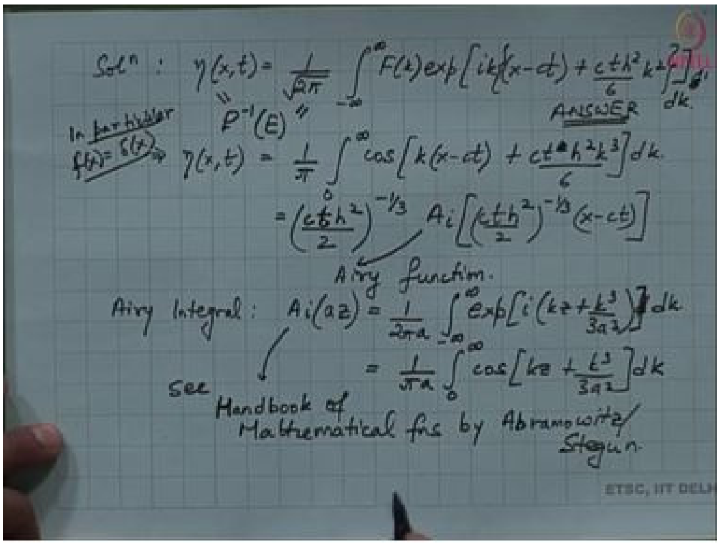
Solution:

$$\text{Assume, } E(k,t) = FT(\eta)$$
$$F(k) = FT(f)$$

So, finally, I apply my Fourier transform to, let us say that this is my equation 1 and this is my equation 2 here. I apply my Fourier transform with respect to x in 1 and 2, I get the following, following expression for E given by

$$E(k,t) = F(k)e^{ikct}\left(\frac{k^2h^2}{6} - 1\right) \tag{3}$$

I get back my initial condition F(k), in the transform plane and so, this solution. So, this is, this expression given by let us say by 3 is the solution to 1 and 2 ok.



So, finally, my solution :

$$\eta(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)\exp\left[ik(x-ct) + \frac{cth^2}{6}k^2\right]dk$$

Then,I have the following expression for the solution. So, then I need to find out this integral. So, then we can see that this integral is an even function. So, notice that I can reduce this integral into the below format.I can reduce it into cos. So, well, I believe. Well, this is the scenario where, you know this is the answer. So, this is what the answer is, I. We cannot proceed ahead before we do not know, before we know the expression for this F.

Now, I am going to talk about, in this next expression I am talking about, a special case. When my function the initial condition is known, let us say initial condition is given by the delta function. So, if that is the scenario; then, I can write my solution to be of the form :

$$\eta(x,t) = \frac{1}{\pi}\int_0^\infty \cos\left[k(x-ct) + \frac{ct^2h^2k^3}{6}\right] dk$$

So, in the case when we know the exact form of this function f the initial condition, I can reduce my solution to, to this expression and this particular expression can be evaluated and this the answer is given as follows:

$$= \left(\frac{cth^2}{2}\right)^{-1/3} A_i\left[\left(\frac{cth^2}{2}\right)^{-1/3}(x-ct)\right]$$

A_i,So, this is also called as the airy function ok. So, in particular, I have the airy integral, the airy integral defined as follows.

$$A_i\left(az\right) = \frac{1}{2\pi a}\int_{-\infty}^\infty \exp\left[i\left(kz + \frac{k}{3a^2}\right)\right] dk$$

you know simplify this further it becomes:

$$= \frac{1}{\pi a}\int_0^\infty \cos\left[kz + \frac{k^3}{3a^2}\right] dk$$

So, this can be found from, this expression can be found from the handbook; see the expression for this airy integral on handbook of mathematical functions, functions by Abramowitz and Stegun. So, this is also given in my course this book, is given in my course description. So, in this book the airy integral and the airy functions are all the form of airy functions are all given ok. So, that completes that, some of the examples that I had to describe for the Fourier transform.