Statistical Inference
Prof. Niladri Chatterjee
Departments of Mathematics
Indian Institute of Technology Delhi

Lecture – 06
Statistical Inference

Welcome students to the 6th lecture of the MOOC series on Statistical Inference. In the last lecture, we were looking at the chi square distribution which is obtained as a square of standard normal variate. If there is one such normal which is x, then we got that x square is distributed as chi square with one degrees of freedom, which is essentially gamma distribution with half and half.

We also found that if we look at the sum of square of n independent normal variates, then that will distribute as chi square with n degrees of freedom which is gamma with half and n by 2. In this lecture, we will look at some more types of random variables in particular.

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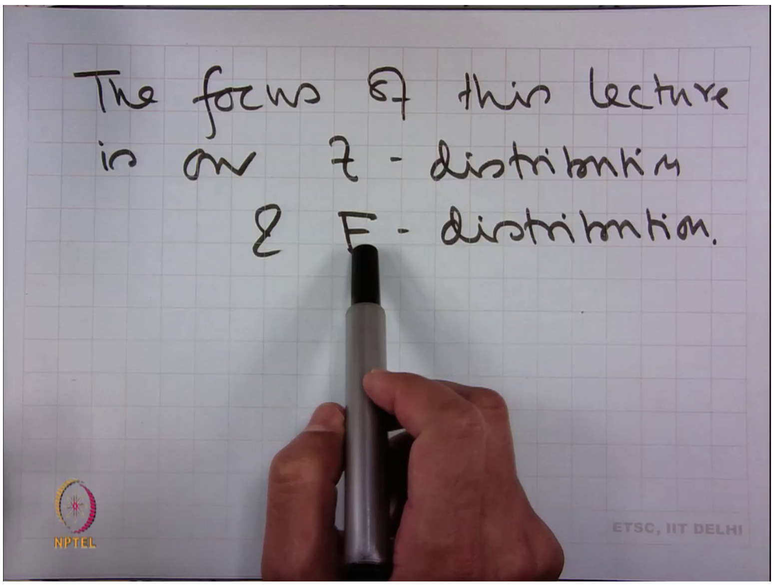

The focus of this lecture is on t distribution and F distribution. These are very important for statistical inference with respect to population.

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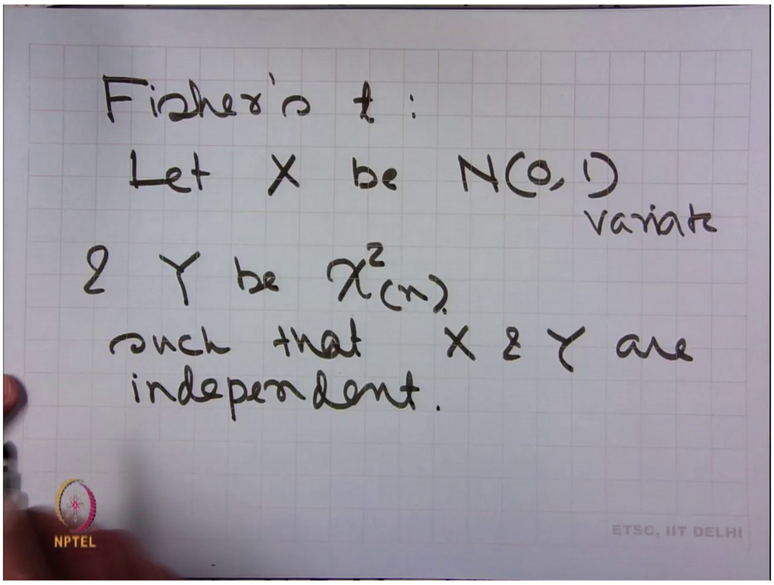

And we will see that how they are used in inferencing about population, but before that in this lecture, I will just derive their pdf. R. A. Fisher's has defined t to be a random variable which is the quotient of 2 independent distribution, where the numerator is a normal 0 1, and the denominator is a function of chi square. So, let me define it first, such that X and Y are independent.

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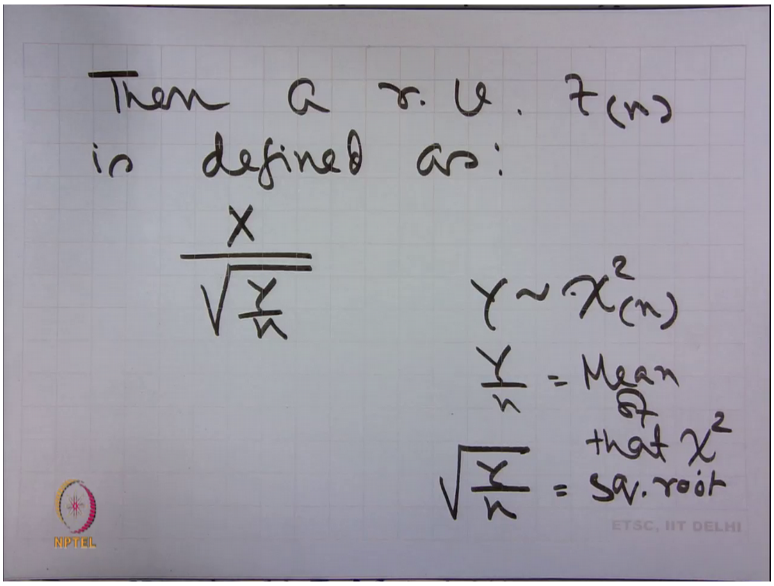

Then a random variable t with n degrees of freedom is defined as X upon root over Y by n. X is a normal variable Y is a chi square variable; therefore, Y by n is mean of that chi square and root over Y by n is equal to square root of that. So, essentially t distribution is the ratio of a standard normal divided by a root mean chi square distribution.

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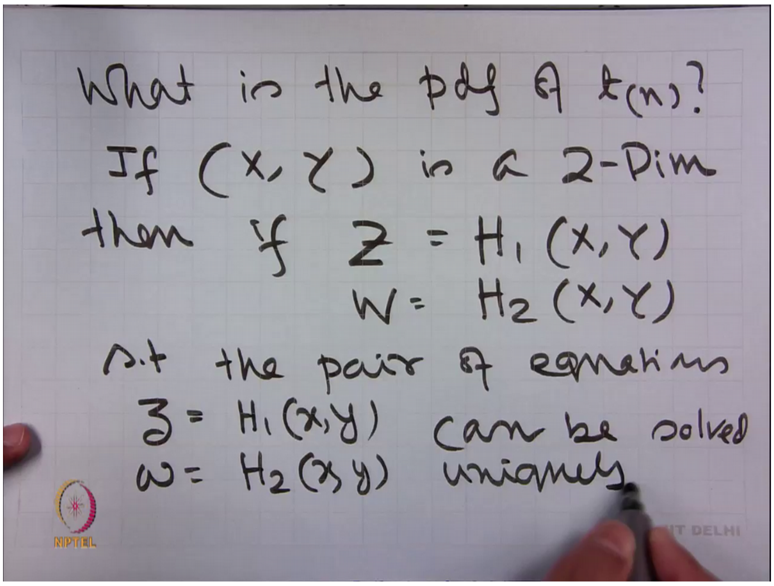

So, what is the pdf of; in order to obtain the pdf of t, I will in order to obtain the distribution of tn, I will appeal to that theorem that we have done in our lecture 5; that if X Y is a 2 dimensional random variable, then if Z is equal to H 1 of X Y and W is equal to H 2 of X Y such that the pair of equations z is equal to H 1 x y and w is equal to H 2 x y can be solved uniquely.

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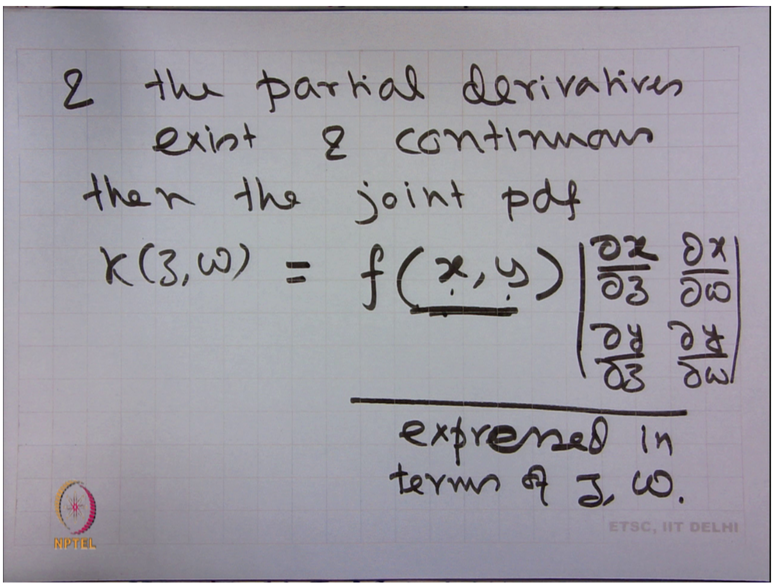

And the partial derivatives exist and continuous, then the joint pdf k z w of z w is given by f of x y multiplied by the Jacobean which is the determinant of and this whole thing expressed in terms of z and w. And that is possible because the transformation is one to one. So, given z w we can uniquely determine the x and y. If we remember this, then we shall use the same theorem for obtaining the pdf of the t distribution with n degrees of freedom.

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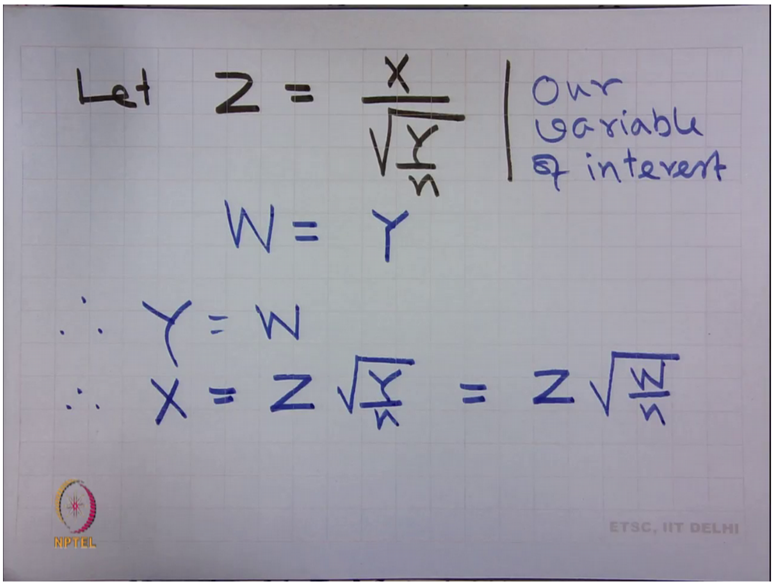

So, let z is equal to X upon root over Y by n. Note that this is our variable of interest. In order to use it we take a dummy variable W is equal to y, ok. Because we need to do transformation to a 2D plane, and then as I have shown with respect to the obtaining the chi square distribution which is summation of 2 independent normal variables square if you remember we have transformed it to R theta then you have integrated out theta.

So, same thing we are going to do here. Therefore, y is equal to W therefore, X is equal to Z into root over Y by n is equal to z into root over W by n. Therefore, given Z W we can uniquely determine X and Y and vice versa.

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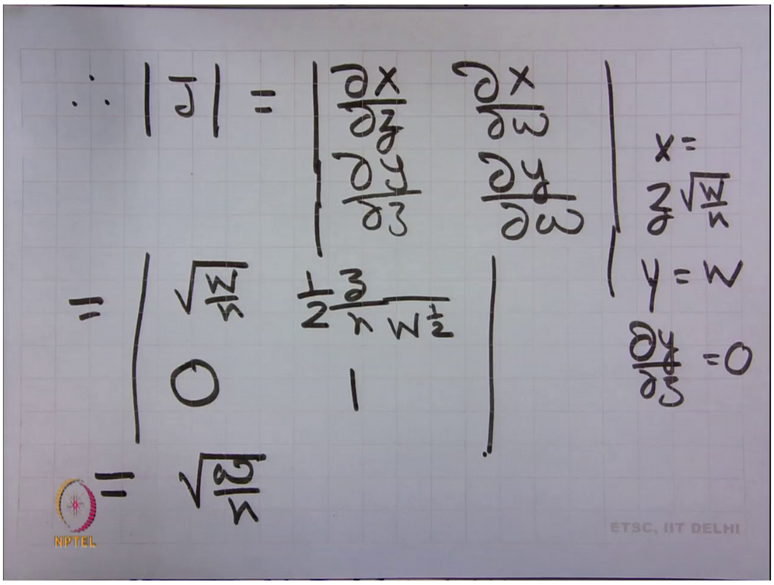

Therefore, what is going to be the determinant or the Jacobean. This is the determinant of del X del Z del X del W del Y del Z and del Y del W; which is equal to x is Z root over W by n. Therefore, del x del z is equal to root over W by n. Del x del w is half z by n z by n into half w to the power minus half. And we know that Y is equal to W therefore, del y del z is equal to 0, and del y del w is equal to 1. Therefore, this determinant is coming out to be root over W by n.

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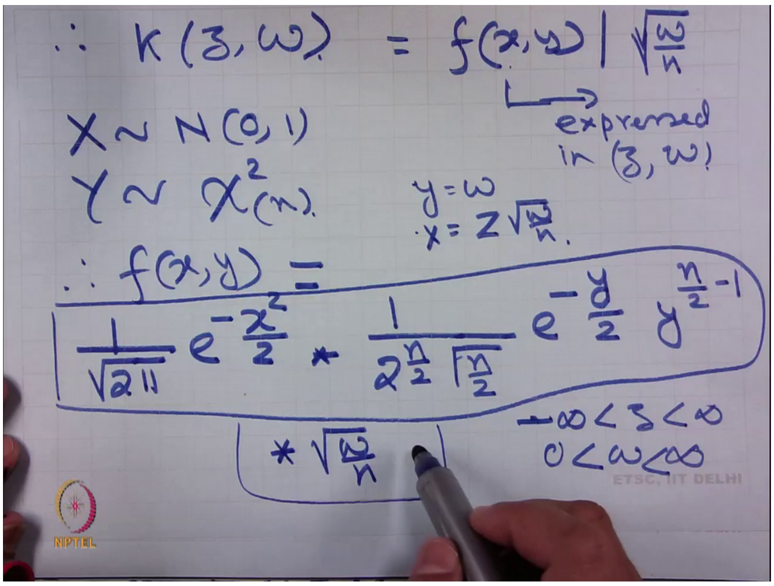

This is important because this will need in obtaining the pdf of z w. Therefore, pdf of z w is f of x y multiplied by the determinant which is w by n expressed in zw. Now x and y are independent, therefore, f of x y is equal to the product of their individual density function, which is 1 over root over 2 pi e to the power minus x square by 2, this comes from x multiplied by y is a chi square with n degrees of freedom. Therefore, it is pdf is lambda power alpha gamma alpha e to the power minus lambda x, x to the power. In fact, x becomes y here, so, e to the power minus y by 2 y to the power n by 2 minus 1.

It is worth noting that z will go from minus infinity to infinity, because z is a standard normal distribution divided by something which is positive, and w goes from 0 to infinity. Now this we have to write in terms of z and w, and our transformation was that y is equal to w and x is equal to z root over w by n. So, we are replacing these things here, and this has to be multiplied by the Jacobean root over w by n. So, this whole thing will give me the pdf of z w.

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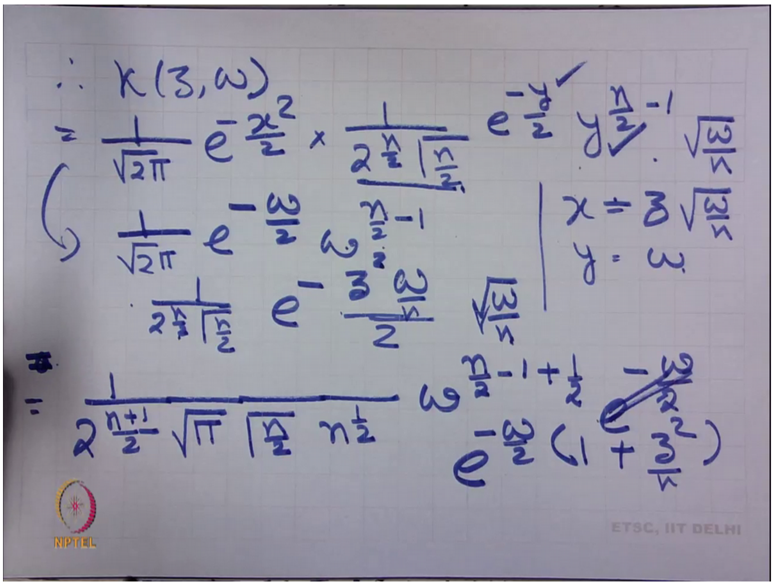

Let me write it again therefore, k z w is equal to 1 over root over 2 pi e to the power minus x square by 2 into 1 over 2 to the power n by 2 gamma n by 2 e to the power minus y by 2 y to the power n by 2 minus 1 into root over w by m. And we have to replace x with z root over w by n and y with w. Therefore, this becomes 1 over root over 2 pi e to the power minus w by 2, w to the power n by 2 minus 1, and from here we get e to the power minus z square into w by n by 2 into root over w by n.

So, this is slightly complicated expression, but this I am doing for you to understand the trick of obtaining the pdf of a function of 2 random variables. So, let us simplify it to some extent what we get? 1 upon 2 to the power n by 2 into half; So, 2 to the power n plus 1 by 2 into root over pi, into gamma or n by 2 into this routine let me take out it is n 2 the power half. So, we have taken out all the constants.

Now, let us look at the variables. So, it is w to the power n by 2 minus 1 plus half multiplied by e to the power minus w by 2. From here it is w by 2 from here also it is w by 2. Multiplied by e to the power minus w by 2 into, from here I am getting 1 from here I am getting 1, from here I am getting z square by n.

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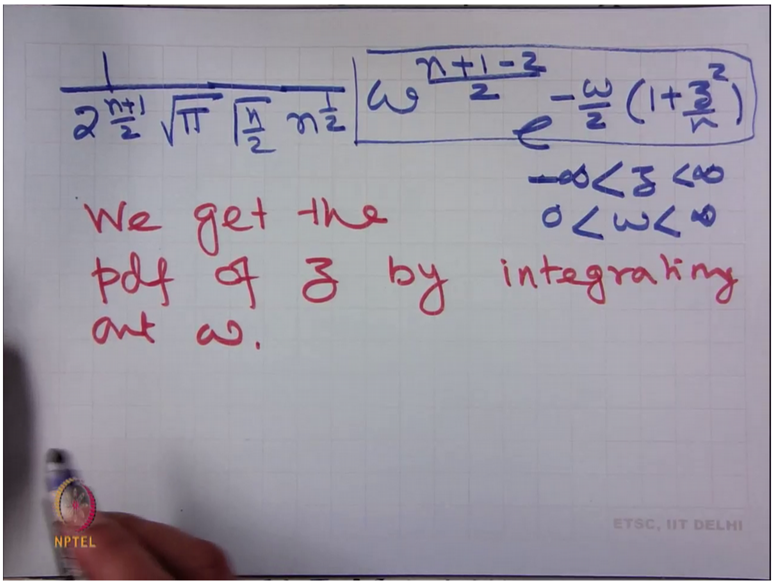

So, this is the joint pdf of z and w, which we can write it as 1 over 2 to the power n plus 1 by 2 root over pi gamma n by 2 n to the power half, w to the power n plus 1 minus 2 by 2, e to the power minus w by 2 into 1 plus z square by n where minus infinity less than z less than infinity, and 0 less than w less than infinity.

Once you obtain the pdf of z and w, we need to find out the pdf of z from there. How do I get it? By integrating out w, right? If we consider this carefully we can see that this actual is coming out to be a gamma integral. So, let us forget about that constant first.

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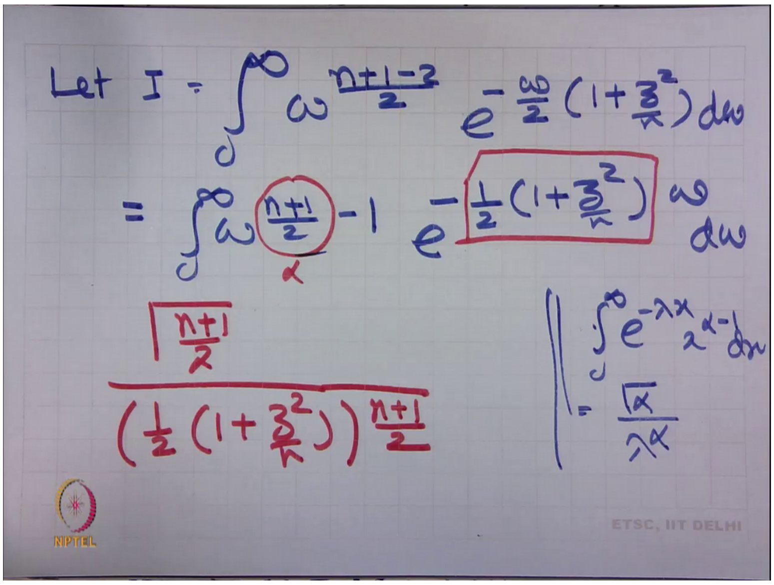

Let us look at this and let us try to find out what is the integral. I am trying to convert it into the gamma form we know that it is e to the power minus lambda x, x to the power alpha minus 1. So, I have put it into something minus 1 into e to the power minus half into 1 plus z square by n w dw. And we know that e to the power minus lambda x into x to the power alpha minus 1 dx that integrates to gamma alpha upon lambda power alpha.

So, we are going to use the same thing. Here alpha is equal to n plus 1 by 2, and here lambda is equal to this whole thing. Therefore, this integration is coming out to be gamma n plus 1 by 2 divided by half into 1 plus z square upon n whole to the power n plus 1 by 2, ok.

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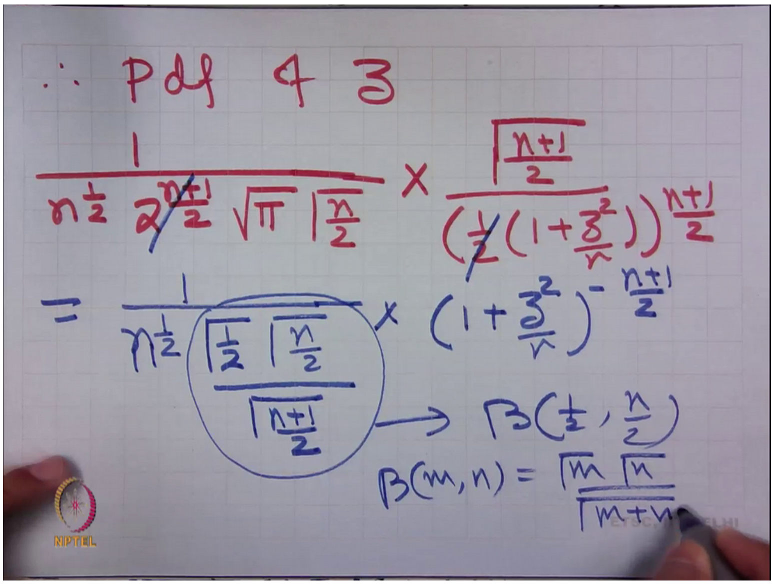

So, this is the integral of this part, therefore pdf of z is first we have to write that constant; which is 1 upon n to the power half 2 to the power n plus 1 by 2 root over pi gamma n by 2. Now, multiplied by gamma n plus 1 by 2 upon half into 1 plus z square by n whole to the power n plus 1 by 2; This 2 to the power n by 2 cancels with this 2 to the power n plus 1 by 2.

Therefore, what we obtain is n 2 the power half root pi we know is that it is gamma half, gamma n by 2 into 1 plus z square by n whole to the power minus n plus 1 by 2. This divided by gamma n plus 1 by 2. This whole thing is gamma half gamma n by 2 upon gamma n plus 1 by 2; actually follows beta half n by 2. Because we know that beta m comma n is equal to gamma m gamma n upon gamma m plus n.

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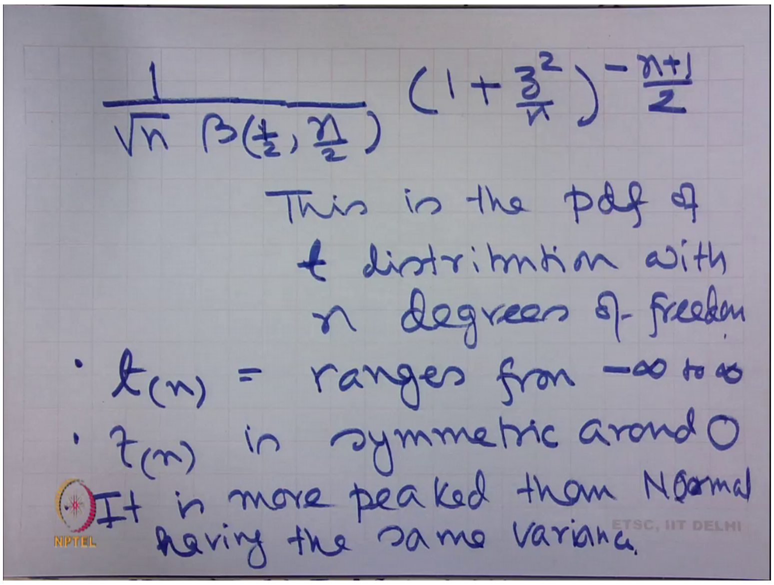

Therefore, this can be written as 1 over root over n, beta half comma n by 2 multiplied by 1 plus z square by n whole to the power minus n plus 1 by 2. So, this is the pdf of t distribution with n degrees of freedom.

Why n degrees? Because in the denominator we had a chi square distribution which is of n degrees of freedom properties tn or t with n degrees of freedom ranges from minus infinity to infinity; tn is symmetric around 0 which is also true for normal. But basically t is more peaked than normal having the same variance. What I mean that the shape of the distribution is symmetric around 0, but if it is shape will vary with the degrees of freedom and therefore, the variance will also keep changing.

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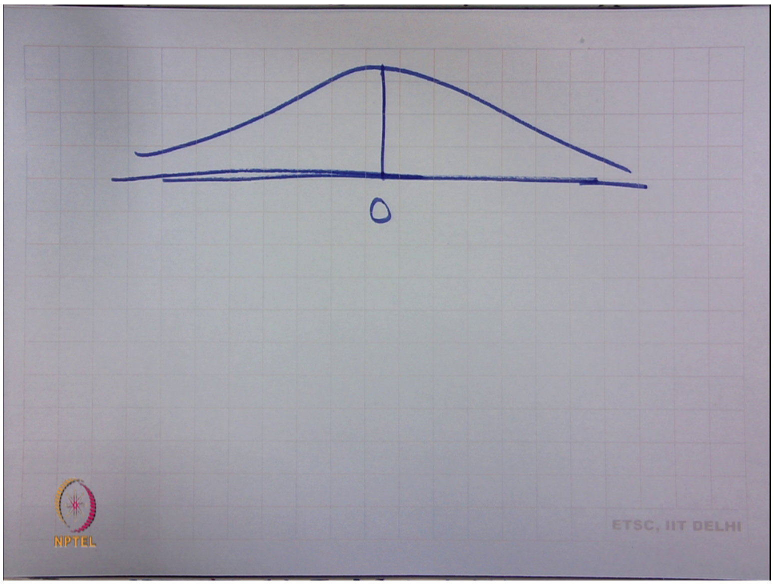

So, if we take a t distribution with certain variance, and consider the corresponding normal distribution with mean 0 and having the same variance, then we will find that t is more peaked than the normal distribution.

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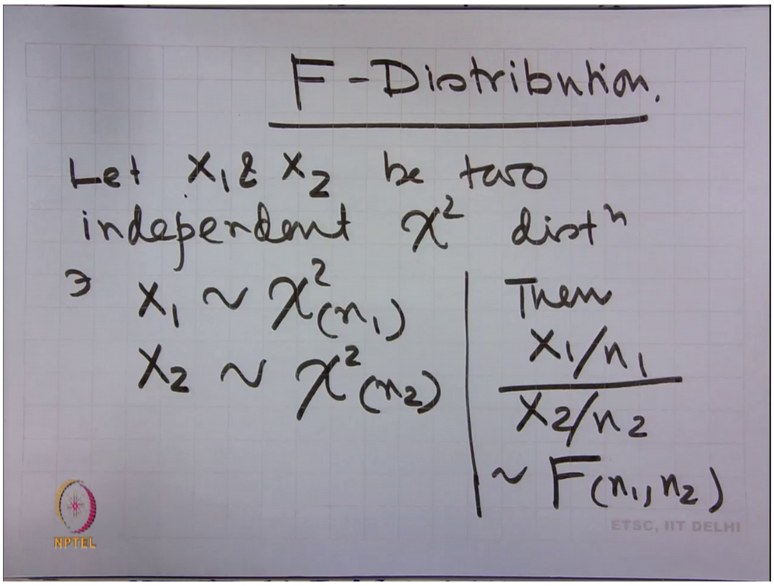

Another important distribution that I like to cover today is called F distribution. What is an F distribution? Let X 1 let X 1 and X 2 be 2 independent chi square distribution; such that X 1 is chi square with n 1 degrees of freedom, and X 2 is chi square with n 2 degrees of freedom. Then x 1 upon n 1 divided by X 2 upon n 2 is called F distribution with n 1 n

2 degrees of freedom. So, basically we are looking at a ratio of 2 mean chi square distribution.

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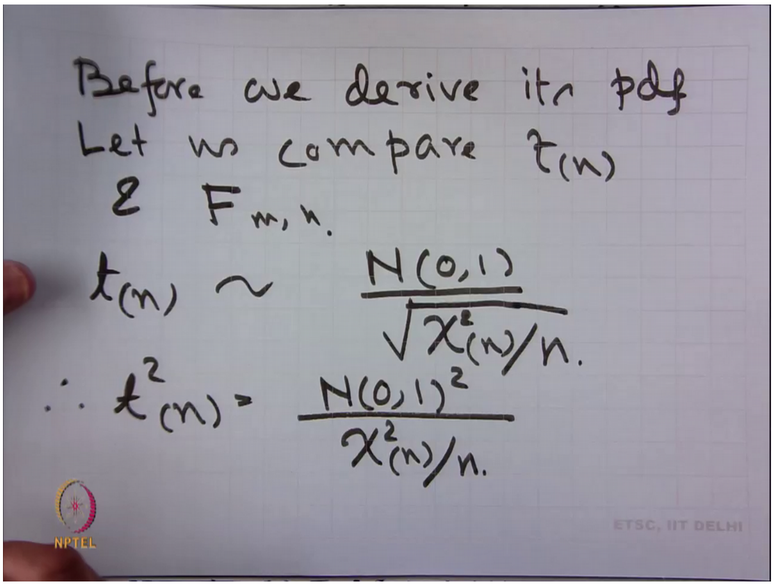

Before I go for let us compare t n and F. T n is defined as normal 0 1 divided by root over chi square n by n. Therefore, t square with n degrees of freedom is equal to normal 0 1 square divided by chi square n by n. We have already seen that normal 0 1 square is basically chi square with one degrees of freedom.

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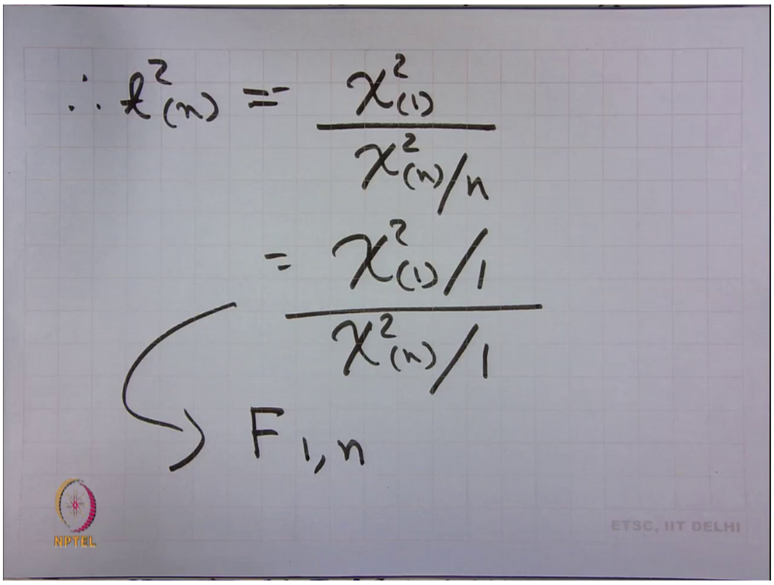

Therefore, t square n is equal to chi square with one degrees of freedom divided by chi square n by n is equal to chi square 1 by 1 divided by chi square n by n. And this is precisely F of 1 and n degrees of freedom. Thus we can see that the square of t distribution also follows an F distribution.

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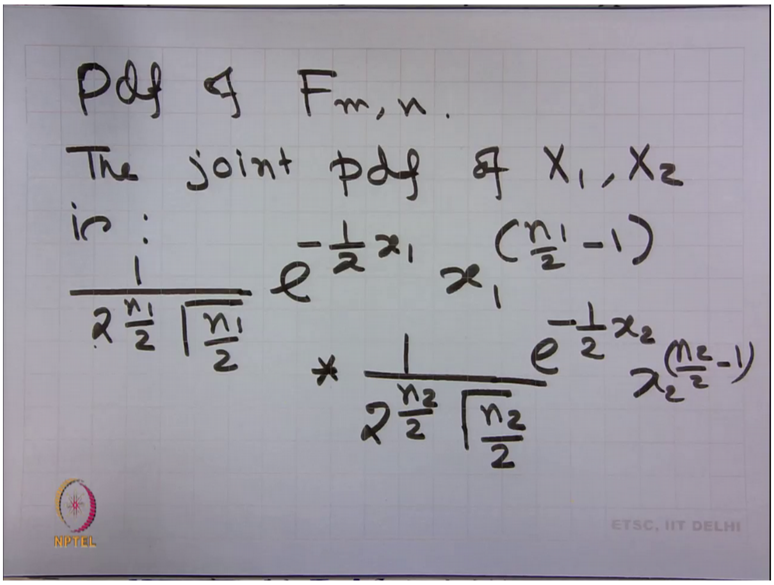

So, what is the pdf? As before, we appeal to the previous theorem. X 1 and X 2 are independent chi square with n 1 and n 2 degrees of freedom. Therefore, their joint pdf is the product of their individual pdf, and we can write it as 2 to the power n 1 by 2 gamma n 1 by 2 e to the power minus half X 1 X 1 to the power n 1 by 2 minus 1 multiplied by 1 upon 2 to the power n 2 by 2; by now you are familiar with the gamma distribution. And chi square with n degrees of freedom is equal to gamma with half and n by 2. So, I am simply writing from there, it is gamma n 2 by 2 e to the power minus half x 2, x 2 to the power n 2 by 2 minus 1.

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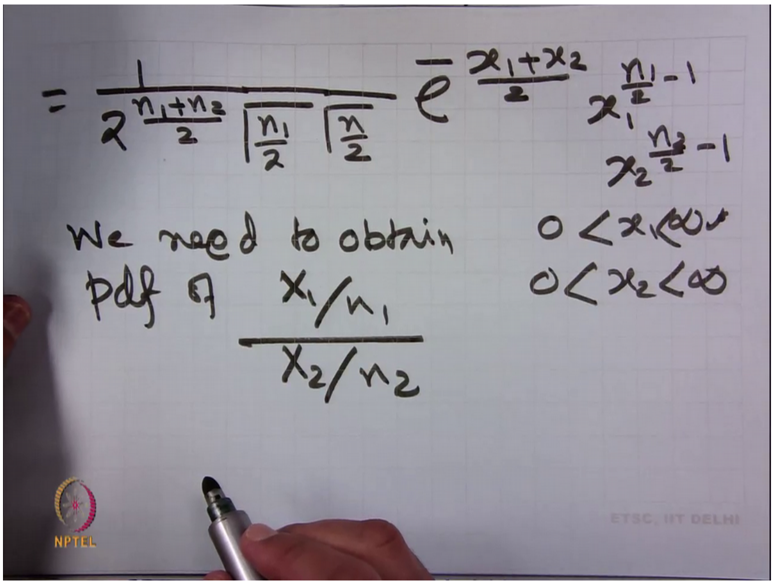

So, that is the joint pdf of x 1 and x 2. So, let me write it as 2 to the power n 1 plus n 2 pi 2, gamma n 1 by 2 gamma n 2 by 2 e to the power minus x 1 plus x 2 by 2 x 1 to the power n 1 by 2 minus 1 x 2 to the power n 2 by 2 minus 1; where 0 less than x 1 less than infinity and 0 less than x 2 less than infinity. So, this is the joint distribution of X 1 and X 2. We need to obtain pdf of X 1 upon n 1 divided by X 2 upon n 2.

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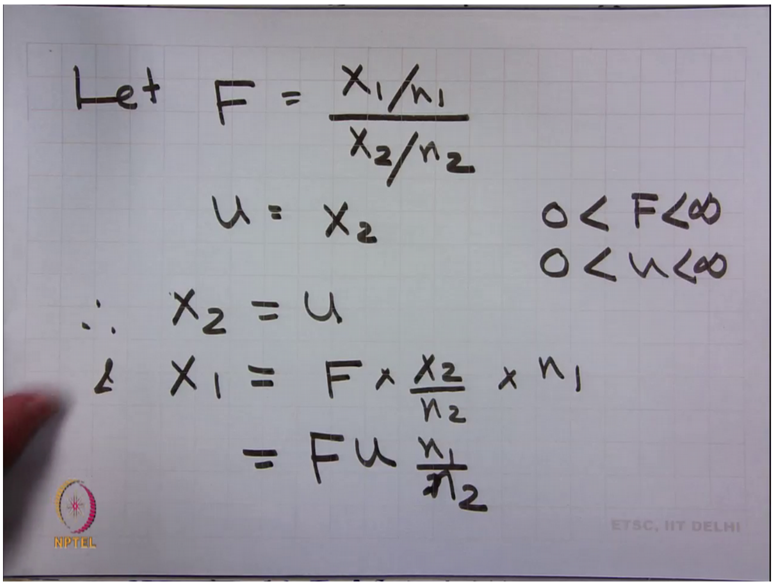

So, let us make the transformation as f is the ratio of 2 chi square both of them are going from 0 to infinity therefore, f is also going from 0 to infinity, and obviously, u is also going from 0 to infinity. Therefore, now we make the inverse transformation. What we get? X 2 is equal to u and X 1 is equal to F u n 1 upon n 2.

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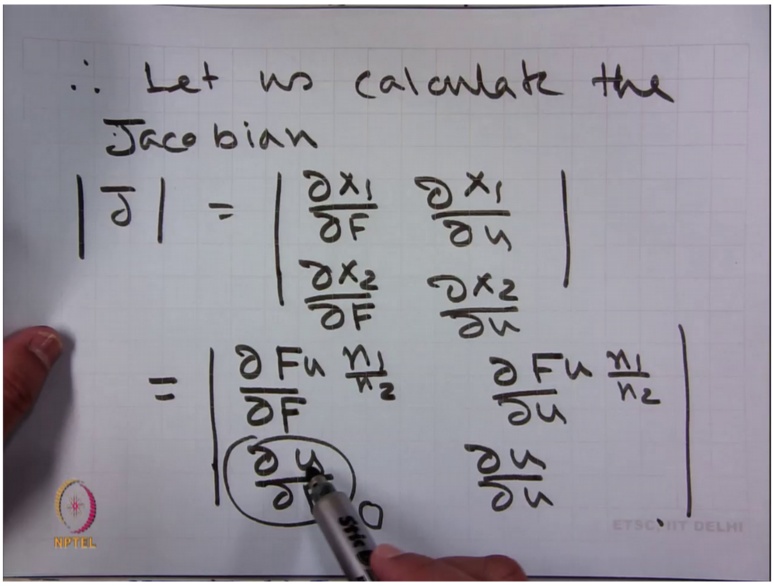

Therefore, let us calculate the Jacobean, modulus of j is equal to del X 1 upon del F del X 1 del u del X 2 del F del x 2 del u is equal to del F u n 1 upon n 2 del F, del Fu n 1 upon n 2 del u, del u del f del u del u, this determinant.

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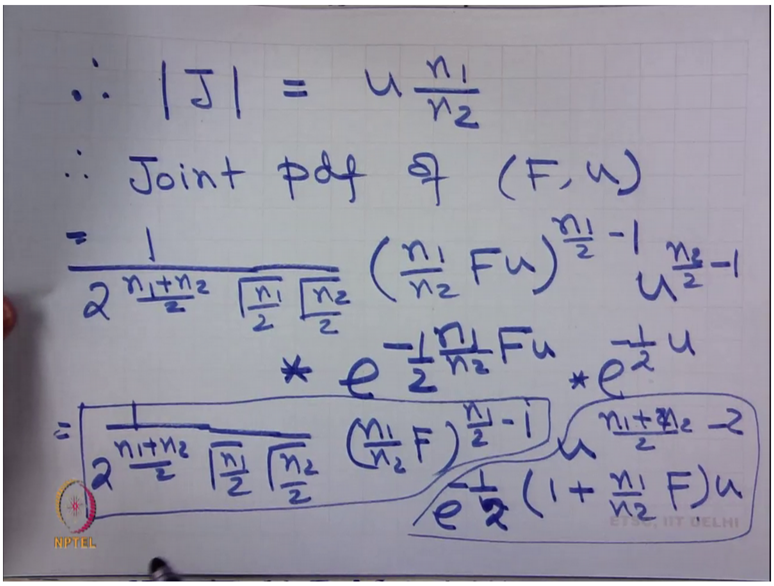

Now, we can see that this is going to be 0, the Jacobean is u n 1 upon n 2, therefore, joint pdf of F u is equal to 1 upon 2 to the power n 1 plus n 2 by 2, gamma n 1 by 2 gamma n

2 by 2 into n 1 upon n 2 F u hold to the power n 1 by 2 minus 1, because we are replacing x with n 1 upon n 2 into F u, multiplied by u 2 the power n 2 by 2 minus 1, multiplied by e to the power minus n 1 upon n 2 fu. Because e to the power minus lambda x. So, e to the power minus half x x is X 1 is n 1 upon n 2 F u into e to the power minus half u.

So, this becomes 1 upon 2 to the power n 1 plus n 2 by 2 gamma n 1 by 2 gamma n 2 by 2 n 1 upon n 2 F to the power n 1 by 2 minus 1, u to the power n 1 plus n 2 by 2 minus 2. E to the power minus half u into 1 plus n 1 upon n 2 f e to the power minus f 1 plus n 1 by n 2 f into u. Therefore, this part is independent of u. And if we look at this part, again it comes e to the power minus some positive constant times u into u to the power n 1 plus n 2 by 2 minus 2 which we can write something minus 1 form.

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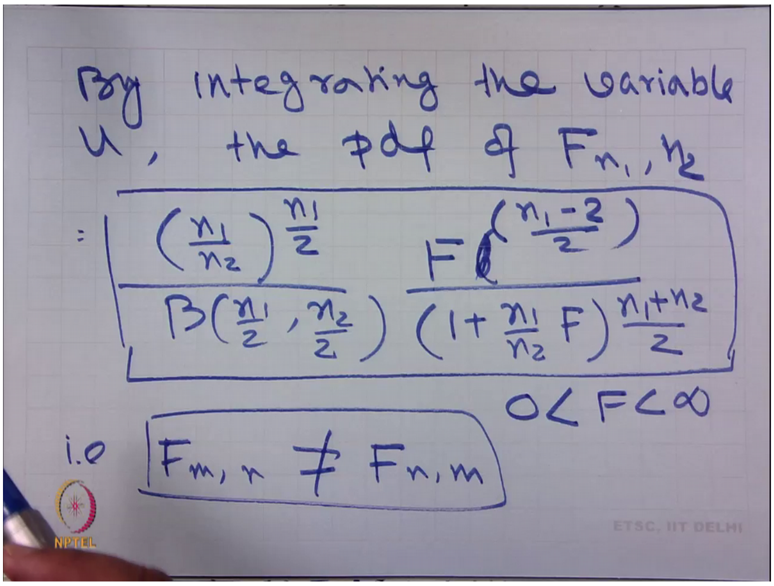

Therefore, this augie again gives rise to a gamma integral, therefore, by integrating it, I am not doing the entire integration, I like you to do it. So, the result is going to be n 1 upon n 2 to the power n 1 by 2, when beta n 1 by 2 n 2 by 2 F F to the power n 1 minus 2 by 2 divided by 1 plus n 1 by n 2, F whole to the power none plus l 2 by 2.

So, this is a fairly complicated expression which gives the pdf of, and if F variate with parameters n 1 and n 2; Note that the order of the 2 constants n 1 and n 2 is very important; that is, F of m n is not equal to F of n comma m. Because F is the ratio of 2 chi square so, depending upon the chi square in the numerator, and the chi square in the denominator the pdf of these will change.

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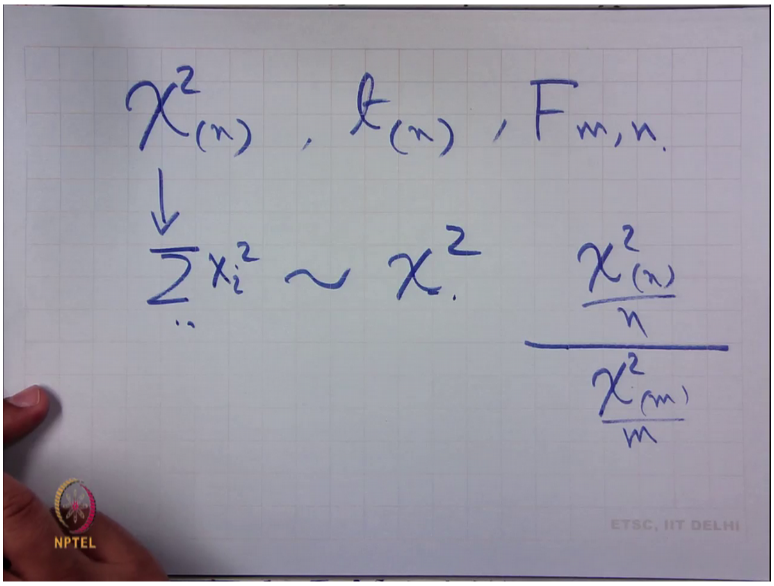

Now, you may ask me, why do we need these failures namely chi square t and F. Let us first look at chi square. Sigma xi square gives rise to chi square. Now suppose x is with mean 0, then variance of x is actually expected value of x square. So, if I consider a sequence of samples from a normal population with mean 0, and we want to find out the sum of it is variants, then we know that the variance of the sum is equal to sum of it is variants because they are independent, and therefore, we do not need their covariance. And therefore, we get to know the distribution of that variance and as you can understand that that will give rise to a chi square distribution.

Now, why do we need F? Suppose we want to compare the variances of two different populations. So, chi square m by n which is a sum of square divided by the divided by the degrees of freedom; so, that gives an idea of the variance of the numerator. Or chi square n by n gives you the idea of the variance of sum of square of normal distribution, or the variance of the population.

Similarly, chi square m by m also gives the variance of another population. So, when we want to compare the variances of 2 population. We know that variance depends upon the unit so; as such the difference between them does not really make sense. The best way to compare them; is by taking their ratio and see what is the relationship between these 2 variances. To compare the variances of two different populations with underlying distribution is normal we need F distribution.

Similarly, we will see later that t can be expressed as a mean upon standard deviation. So, from there we can look at the interesting property of the underlying population when we use t distribution. So, all three of them have come from normal distribution; n actually means the sample size. So, how many samples you are taking from a normal population? And then when we want to compare when we want to infer about one particular population or want to compare between 2 populations we will be using these random variables in an effective way. We shall discuss that in my subsequent lectures.

Thank you.