Introduction to Probability Theory and Stochastic Processes

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Lecture – 92

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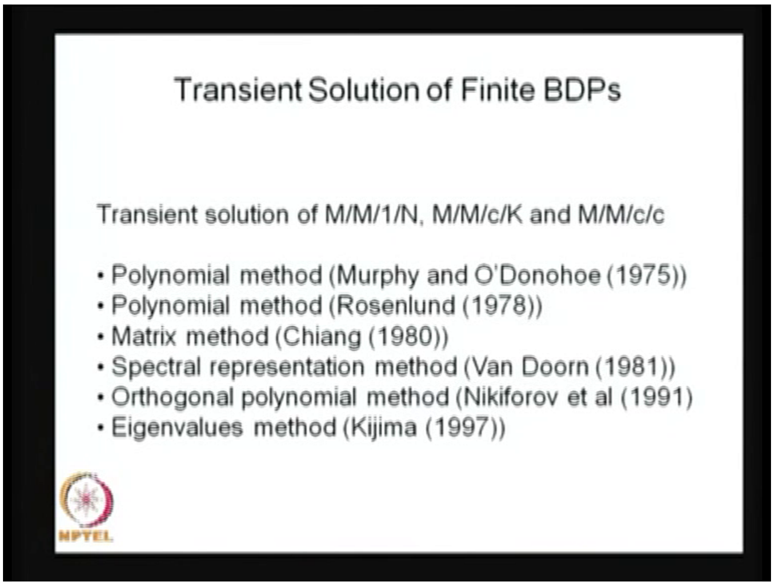

Now, I am explaining the transient solution of a finite birth death process. So, using this one can find out the transient solution of the birth death process which I have discussed the today's class M M 1 N, M M c K and M M c c also. So, the logic is same that means, you have a birth death process with the finite state space therefore the q matrix is going to be a degree whatever be the number of states in the state space.

And it is going to be a tri diagonal matrix and, you know the lambda ns and mu ns birth rates as well as the death rates. And the birth rates and the death rates are going to be different for the these three models.

There are many literature over the transient solution of a finite birth death process started with the Murphy and O'Donohoe he uses the polynomial method and 1978 Roseland also found the transient solution for the finite BDP using again different polynomial methods. And Chiang 1980 he made a matrix method to get this a transient solution.

Then later Van Doorn gave the solution using a spectral representation method and, Nikiforov et al 1991, he also gave the transient solution using orthogonal polynomial.

And the later Kijima also gave the solution using eigenvalues methods. So, these are all the literatures for getting the transient solution of a finite birth death process.

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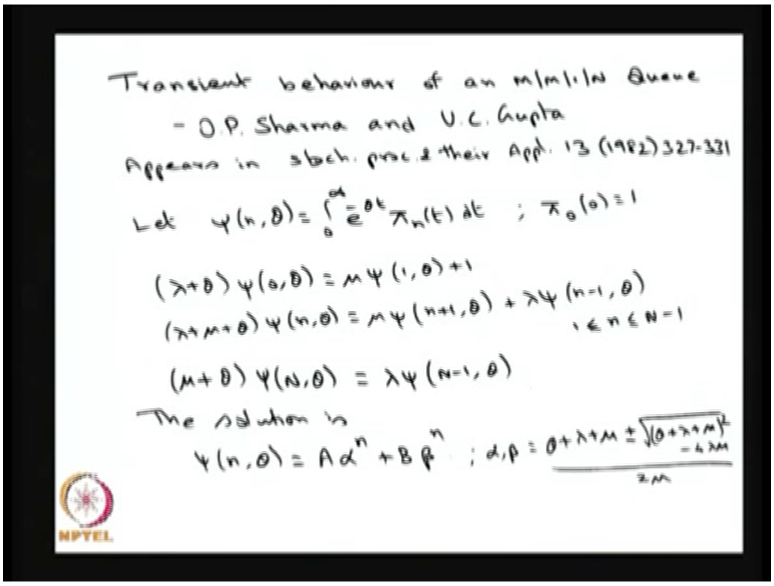

And here I am going to explain how to get the transient behaviour of M M 1 N queue in a very simplest form.

Even though there are this many literature and many more literature for the finite birth death process, but here I am explaining the overview of how to get the transient behaviour of a M M 1 N queue. And, this is by O P Sharma and U C Gupta, it appears in a stochastic processes and their applications, volume 13 1982.

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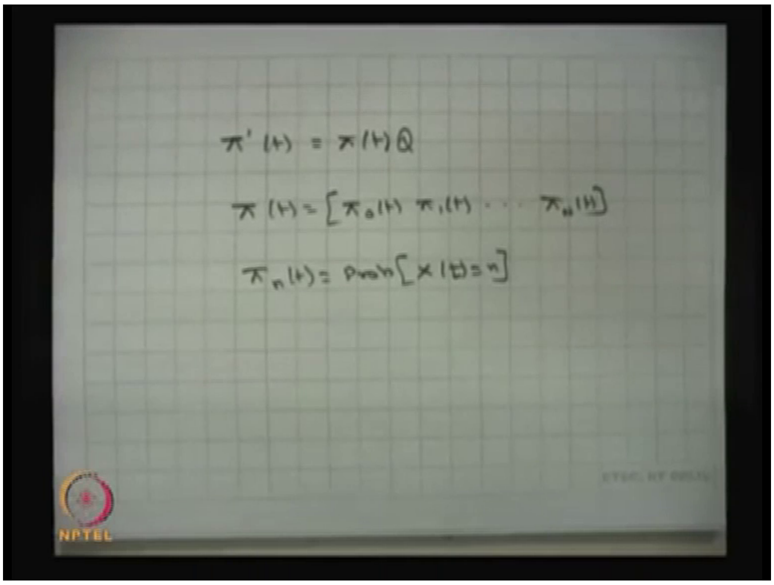

So, what this method work, you start with the forward Kolmogorov equation, that is a pi t pi dash t that is started with the pi dash of t, that is equal to pi of t into Q matrix.

Where pi is the matrix and pi dash is the, with the derivatives and the Q is the infinitesimal matrix take a forward Kolmogorov equation. Then use the Laplace transform for each pi n of t you take the sorry here, the pi dash of t is a vector it is a distribution of a X of t therefore, this is a vector and this is a vector and Q is the matrix not the matrix which is (Refer Time: 03.26). So, this is a vector and this is a vector and t Q is the matrix.

So, take a Laplace transform for each probability, where the pi n of t that is nothing but so, the pi of t is a vector that started with the pi naught of 0 t pi 1 of t and so, on pi n of t where pi n of t is nothing, but what is the probability that the same notation I started, when I discussed the birth death sorry continuous time Markov chain, what is the probability that n customers in the system at time t.

It is an unconditional probability, distribution so, pi n of t is probability that n customers in the system at time t and, using pi n of t you get the vector and you make a forward Kolmogorov equation pi dash of t is equal to pi of t times Q.

And take a Laplace transform for each pi n of t, that exist, because this is a probability and the conditions for the Laplace transform of these function satisfies, you can cross check therefore, you are taking a Laplace transform, this is going to be a function of theta.

Before taking a Laplace transform you need a initial condition also. So, at time 0 you assume that no customer in the system, at time 0 no customer in the system that means, the X of 0 is equal to 0 therefore, that probability is going to be 1 and all other probabilities are going to be 0 that is the initial probability vector.

So, use this initial probability vector and, apply it over the forward Kolmogorov equation, taking a Laplace transform you will get the system of algebraic equation. Since you are using the pi naught of 0 is equal to 1, you will get the first equation with the term 1 and all other terms are going to be 0. And you know the Laplace transform of derivative of the function so, you substitute you take a Laplace transform over the forward Kolmogorov equation, with this initial condition as well as pi ns of 0 is equal to 0 for n naught equal to 0.

So, you will have a algebraic equation that is a n plus one algebraic equations it is a function of theta. You have to solve this algebraic equation system of algebraic equation in terms of theta, once you are able to solve these and take a inverse Laplace transform and that is going to be the system size at any time t. You can start saying that this is going to be of the solution A times alpha n and B times beta power n, where alpha and beta are given in this form.

Where alpha is equal to this plus something and then beta is equal to minus something, minus square root of this expression. So, you will you have alpha as well as beta. Now, what do you want to find out, if you find out the constant A, A and B, you can get the Laplace transform of a pi n of t, then you take a inverse Laplace transform and you get the pi n of t.

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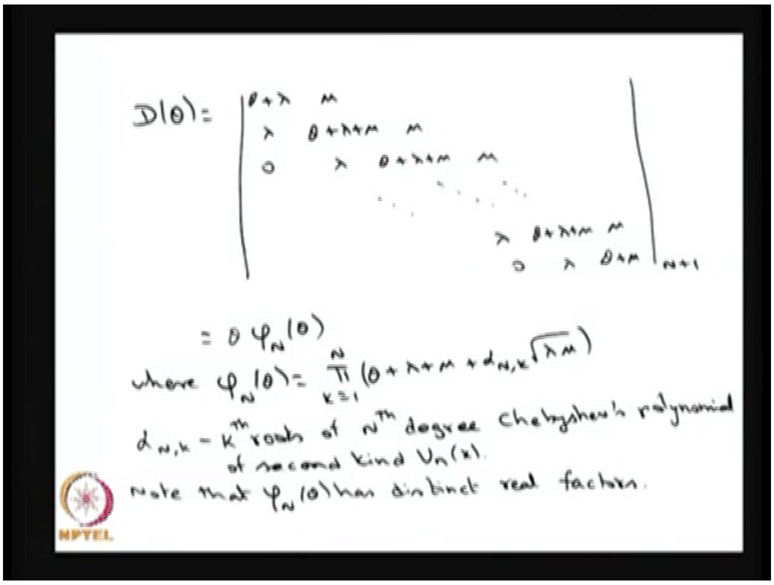

So, for that you need the determinant of matrix of this form, and here this is nothing, but all these values are death rates and these are all the birth rates and this is corresponding to the M M 1 N model. And the same logic goes for the transient solution of a M M c K as well as M M c c.

So, instead of this a lambdas and mu s you will have a corresponding birth rates, and the death rates , but ultimately you will have a N plus 1 matrix determinant as a function of theta. And since these three models are going to be a irreducible, positive recurrent the stationary probability and limiting probabilities exist.

Therefore, this determinant is going to be always of the form theta times, some other function as a degree as a polynomial of degree N in a function of theta. So, this theta is corresponding to the stationary probabilities, or the limiting probabilities therefore, always you can get the N plus 1 th degree matrix 1 th order matrix determinant that is theta times the polynomial of degree N as a function of theta.

For the M M 1 N model the birth rates are lambda and the death rates are mu and, he can get this polynomial also in the form of product. The product of a theta plus lambda plus mu times alpha of N comma K square root of lambda mu, where alpha of N comma K is nothing, but the K roots of Nth degree Chebyshevs polynomial of second kind.

There is a relation between the birth death process with the orthogonal polynomial, for instant the M M 1 N model the finite capacity M M 1 N model, the corresponding orthogonal polynomial for this birth death process is the Chebyshevs polynomial of the second kind.

Similarly, you can say the orthogonal polynomial corresponding to the N M c c model that is a Charlier polynomial like that we can discuss the orthogonal polynomialcorresponding orthogonal polynomial for the finite capacity birth death processes.

So, here for the M M 1 N model this is related to the Chebyshevs polynomial of second kind that is U n of X. So, once you are able to get the Chebyshevs polynomial roots and that is roots is going to play a role in the product form and that is going to be the polynomial, note that this polynomial has a distinct real factors.

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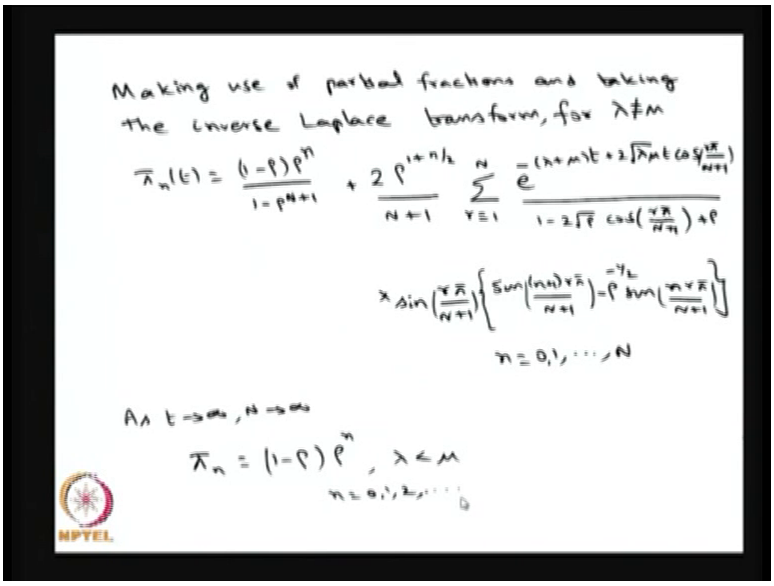

Therefore you can use the partial fraction, then you take a inverse Laplace transform finally, you can get the pi n of t, I am skipping all the simplification part.

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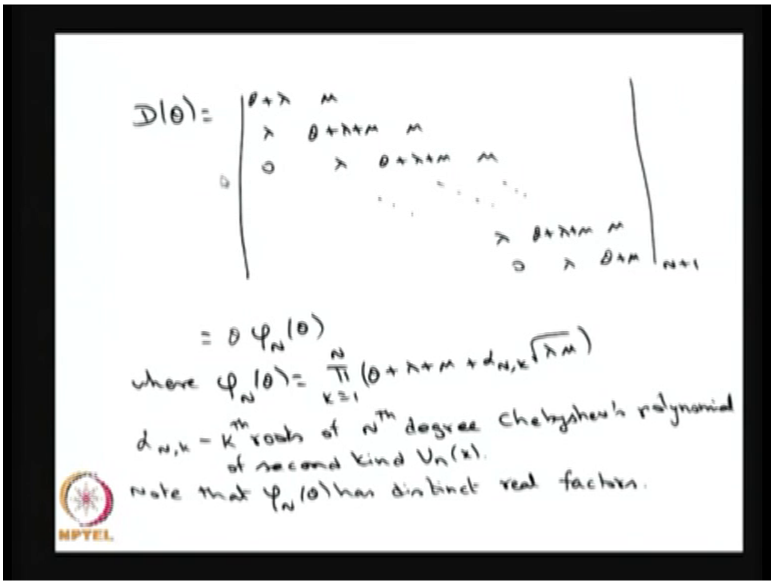

And the main logic is this N plus 1th order matrix determinant and, that determinants has the factors and those factors are related to the Chebyshevs polynomial roots.

So, once you use all those logics and use the partial fraction then finally, you take a inverse Laplace transform, for lambda is not equal to mu you will get steady state, or stationary probabilities plus, this expression and this is a function of t, e power minus lambda plus mu times t plus 2 times square root of lambda mu times t cos of r by pi n plus 1 and denominator this expression multiplied by this and, here this result is related to the initial condition 0 that means, at time 0 the system is empty.

If the system is not empty then you will have a one more expression here, sin of this minus another term. So, that is you will have a little bigger expression for system size is a not empty and the these theta times this that will give the corresponding partial fraction and so, on inverse Laplace it will give a the terms which is independent of t and, that is a related to the steady state probabilities. Because if you put t tends to infinity this quantities are greater than 0.

So, as a t tends to infinity the whole terms will tends to 0 therefore, as a t tends to infinity will have a pi n of t is equal to this expression and, this is valid for rho is less than 1, with that condition rho is less than 1, with the condition rho is less than 1, those terms will tends to 0 and you will have only this term and that is going to be the steady state, or limiting probabilities for M M 1 N model.

If you make also N tends to infinity along with the t tends to infinity, you will have pi ns that is the steady state probability for the M M 1 infinity model. So, even though I have explained a M M 1 N transient solution in a brief way.

but the same logic goes for the M M c c model also the only difference is this determinant has the lambdas and instead of mus, you will have mu 2 mu 3 mu n and so on. And instead of the Chebyshevs polynomial you land up with the Charlier polynomial, but there is a difference between this M M 1 N model and the M M c c model transient solution.

Since the Chebyshevs polynomial has a closed form roots, you can find out the factors so, here these are all the factors and you know the factors as well as you can get the closed form expression for the M M 1 N transient solution whereas, a Charlier polynomial does not have a closed form roots. Therefore, you land up with the numerical result for the transient solution for M M 1 M M c c model application of a continuous time Markova chain.

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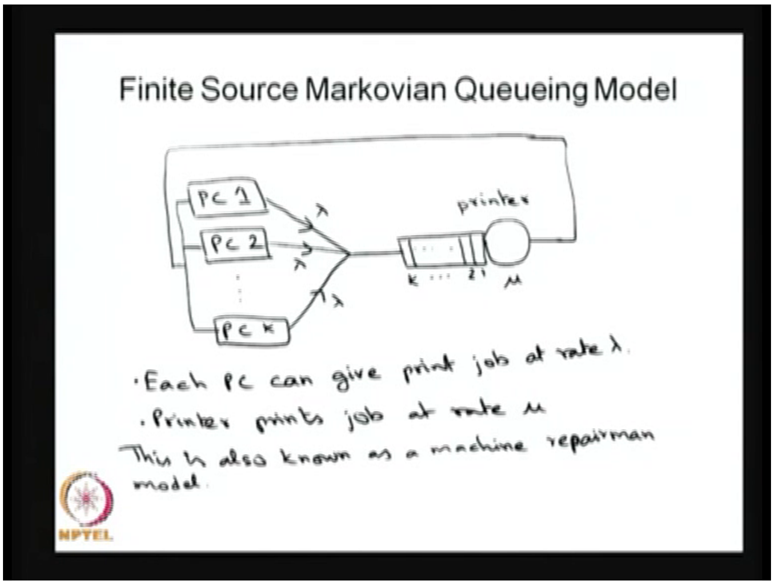

That is a finite source Markovian Queueing models. This model is also known as a machine repairman model and, you can think of this PCs are nothing, but the machines and this is nothing, but the repairman and, here the scenario is we have a K PCs and, each PC can give a print job.

And the inter arrival of a print jobs that is exponentially distributed by the each PC therefore, the print jobs that is follow a arrival process, that is the Poisson process with the parameter lambda from each PC. And, once the print jobs comes into the printer it will wait for the print.

And the time taken for the each print that is also exponentially distributed with the parameter mu and, here there is another assumption, before the first print is over by the same PC it cannot give the another print command. Therefore, after the print is over by any one particular print job of any PC, then this things will go back to the same thing, then with the inter arrival of print jobs generated that is exponentially distributed, then the print job can come into the printer.

So, with these assumptions you can think of the stochastic process; that means, a number of print jobs at any time t in the printer, that is going to form a stochastic process and, with the assumption of inter arrival of a print jobs that is exponential and, actual printing job that is exponentially distributed and so, on.

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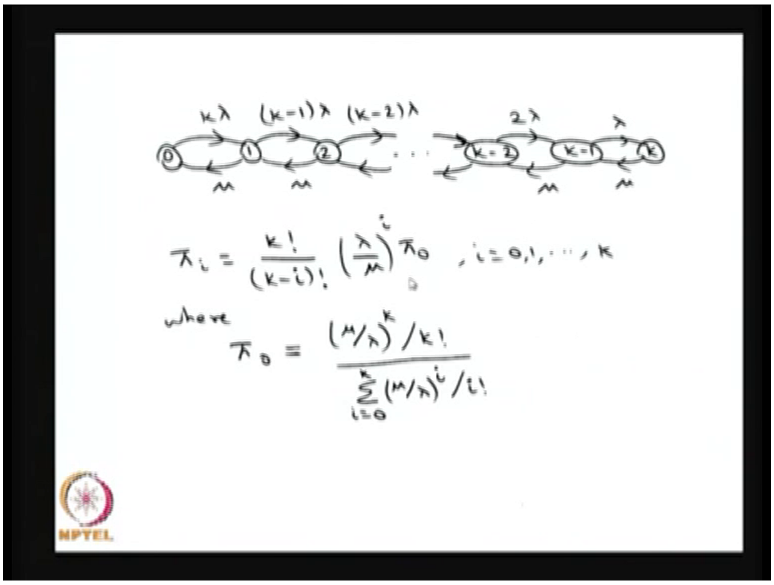

Therefore this is going to be a birth death process, with the birth rates or k times lambda and k minus 1 times lambda and so on. Whereas, the death rates that is mu because we have only one repair. So, this is nothing, but system size, number of jobs in the print job printer.

So, therefore, that varies from 0 to capital K, because we are making the assumption more than one print job cannot be given by the same PC before the print is over. And from 0 to 1 the arrival rate will be any one of the k PCs therefore, the arrival rate is k times lambda and, already one print job is there in the system printer therefore, out of k minus 1 PCs 1 print job can come therefore, the inter arrival time that is exponentially distributed with the parameter k minus 1 times and lambda and so on.

So, this is the way you can visualize the birth rates whereas, the death rates are mu. Once you know the birth rates and the death rates you can apply the birth death process concept to get the steady state probabilities. So, here we are getting the pi is in terms of pi naught and the using summation of pi is equal to 1 you are getting the pi naught also. And once you know the steady state probability you can get the all other measures.

So, the difference is in this model it is a finite source therefore, the birth rates are the function of its a state dependent birth rates whereas, the death rates are mus only.

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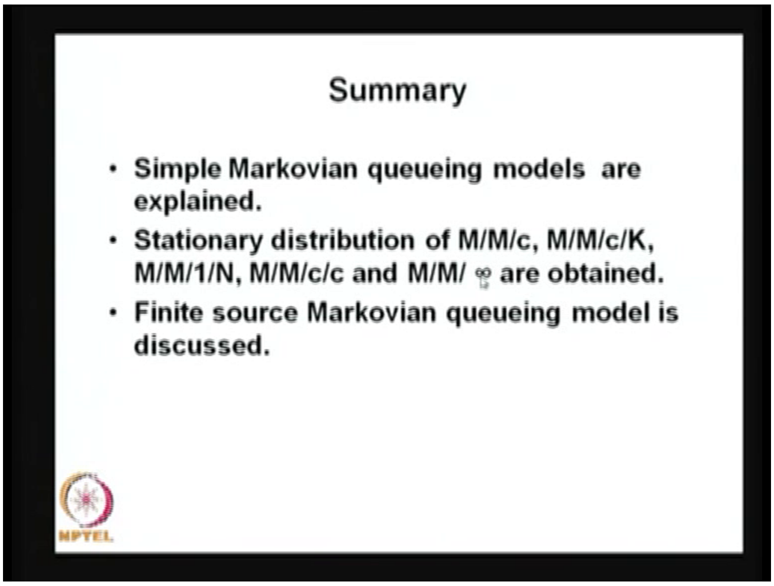

Simulation of a queueing model I will do it in the next lecture the summary of today's lecture, I have discussed the simple Markovian queueing models other than mm one infinity that, I have discussed in the lecture previous lecture. And the stationary distribution and the all other performance measures using the birth death process, we have discussed for these queueing models. And finally, I discussed the finite source Markovian queueing model also.

Thanks.