Introduction to Probability Theory and Stochastic Processes
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Lecture – 89

In this lecture we are going to consider the other simple Markovian queueing models as a application of a continuous time Markov chain.

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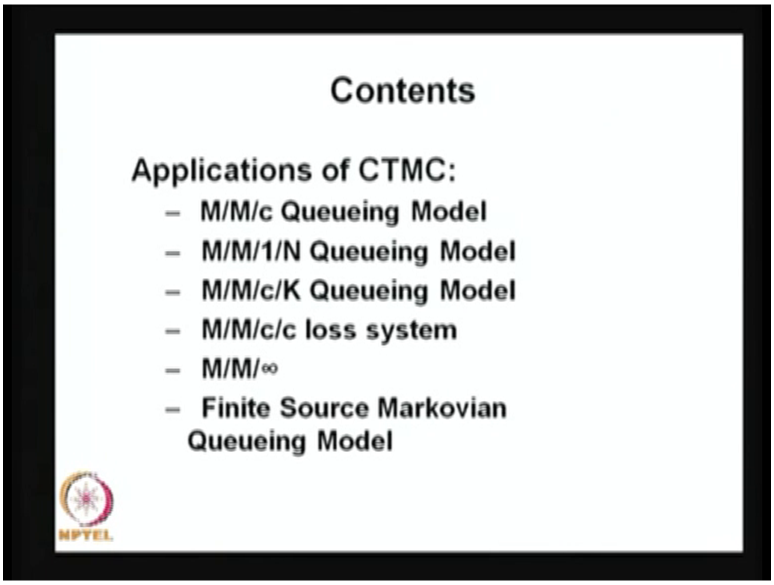

So, in this lecture I am planning to discuss other than MM1 queueing model, I am going to discuss the simple Markovian queueing models, starting with MMc infinity queueing model, then the finite capacity model Markovian set up MM1 n queueing model.

Then I am going to discuss the multi-server finite capacity model, that is MMcK queueing model. After that I am going to discuss the last system that is MMcc model. For a infinite server model that is MM infinity also I am going to discuss. At the end I am going to discuss the finite source Markovian queueing model also. Whereas, the other 5 models the population is infinite source. So, the last one is a finite source Markovian queueing model also I am going to discuss as the application of continuous time Markov chain.

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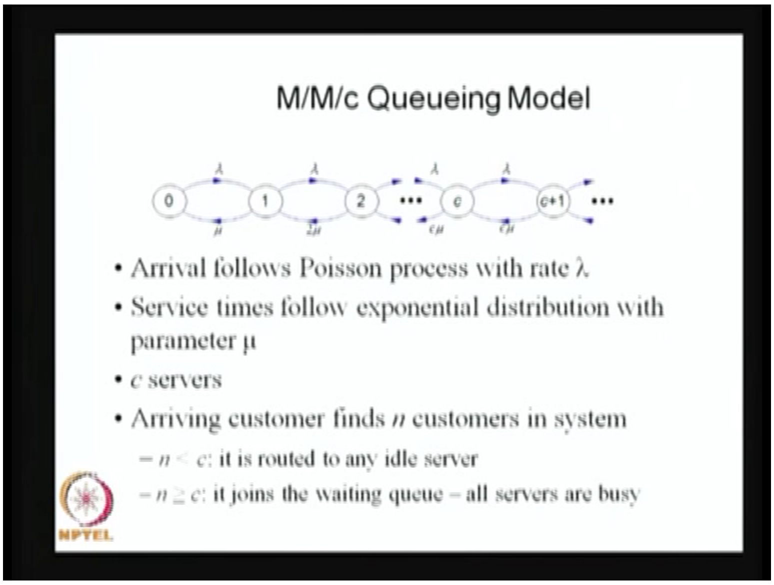

The first model is a multi-server infinite capacity Markovian queueing model. The letter m denotes the inter arrival time is exponentially distributed with parameter lambda. The service time by the each server that is exponentially distributed with the parameter mu. And all we have more than one servers. Suppose you consider as a c, where c is a positive integer and all the servers are identical.

And each server is a during the service which is exponentially distributed with the parameter mu which is independent of the all other servers. And the service time is independent with the inter arrival time also. With these assumptions, if you make a random variable x of t is the number of customers in the system at any time t that is a stochastic process.

Since the powerful values of number of customers in the system at any time t that is going to be 0, 1, 2 and so on. Therefore, it is a discrete state and you are observing the queueing system at any time t therefore, it is a continuous time; so, discrete state continuous time stochastic process. And if you observe the system is a keep moving into the different states because of either arrival or the service completion from the any one of the c servers.

So, suppose there are no customer in the system in the system moves from the state 0 to 1 by one arrival. So, the inter arrival time is exponentially distributed therefore, the rate in which the system is moving from the state 0 to 1 is lambda. Like that you can visualize the rates for system moving from 1 to 2 2 to 3 and so on whereas, whenever the system size is 1 2 and so on till c. Since we have a c number of servers in the system, who are entering into the system they will get they will start getting the service immediately.

Suppose the system goes from state 1 to 0 that means the customer enter into the system and he get the service immediately. And the service time is exponentially distributed with the parameter mu. Therefore, whenever the service is completed, the system goes from the state 1 to 0 therefore, the rate is mu. Whereas, from 2 to 1 there are 2 customers in the system and both are under service. At any time in if any one of the servers completely service, then the system moves from 2 to 1.

So, the service completion will be minimum off a the service time of the both the server. Since in each server is doing the service exponentially distributed with the parameter mu therefore, the minimum of 2 exponential and both are independent also. Therefore, that is also going to be exponentially distributed with the sum of parameters. So, it is going to be parameter will be mu plus mu that is 2 mu.

So, the system moves from the state 2 to 1 will be the rate will be 2 mu. Like that it will be keep going till the state from c to c minus 1; that means, we have c servers therefore, whenever the system size is also less than or equal to c; that means, all the customers are under service.

Now, we will discuss the rate in which the system is moving from the state c plus 1 to c. This is some state is a c plus 1; that means, an the number of customers in the system that is c plus 1. We have c servers therefore; one customer will be waiting for the service waiting in the q.

Therefore, in the system is moving from c plus 1 to c that is nothing but one of the server completed the service out of c servers therefore, the serve the rate will be the service time completion service time will be exponential distribution with the parameter c mu.

Not c plus one mu it is a, we have only c servers therefore, the minimum of exponentially distributed with the parameters mu and so on with the c exponentially distributed random variables. Therefore, that is going to be exponential distribution with the parameter mu plus mu plus there are c mu therefore, it is going to be c mu. Like that the rate will be the death rate will be c mu after c plus 1 onwards.

Whereas from 0 to c it will be mu 2 mu 3 mu and so on, till c mu after that it will be a c mu, from the state from c c plus 1 to c c plus 2 to c plus 1 and so on. And if you see the state transition diagram, you can observe that this is a birth death process. So, before that if let me explain what is a MMc infinity means whenever c customers or c servers or any one of the c servers are available then the customers will get the service immediately. If all the c servers are busy, then the customer has to wait till any one of the c servers are going to be if completing their service.

So, that is the way the system works therefore, you will have the system size.

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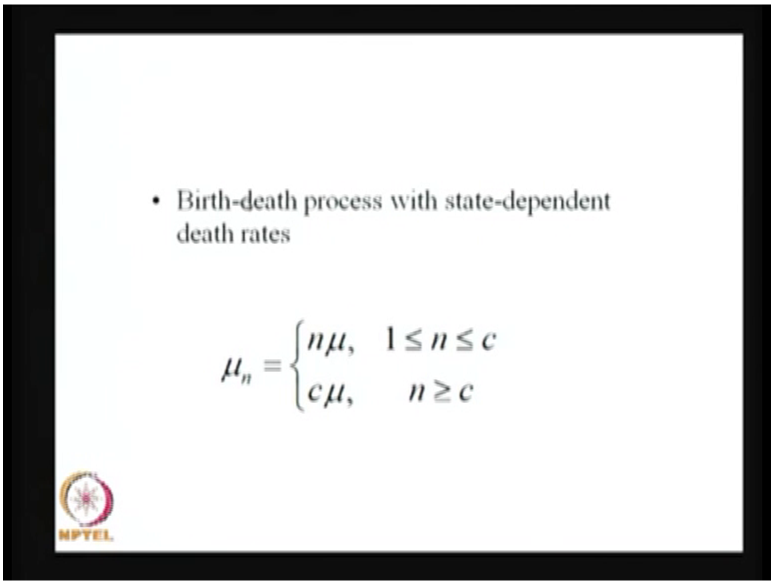

The system size the underlying stochastic process is going to be a birth death process as a it is a special case of a continuous time Markov chain. Because the transitions are only the neighbors transition with the forward rates that is lambda and backward rates are the death rates are going to be mu 2 mu and so on.

Therefore, this is a special case of a continuous time Markov chain, the underlying stochastic process for the MMc infinity model. That is a birth death process the birth rates are lambda. Whereas, the death rates depends on the n, the mu n is a function of n therefore, it is called a state dependent and death rates. It need not be the function n times mu, it can be a function of n then we can use the word state dependent. So, here it is a linear function.

So, state dependent death rates and the death rates are n times mu whenever n is lies between 1 to c, and the mu n is going to be c times mu for n is greater than or equal to c that you can observe it from the state transition diagram also. The death rates are going to be c mu here also c mu and so on therefore; this is a birth death process with the state dependent death rates.

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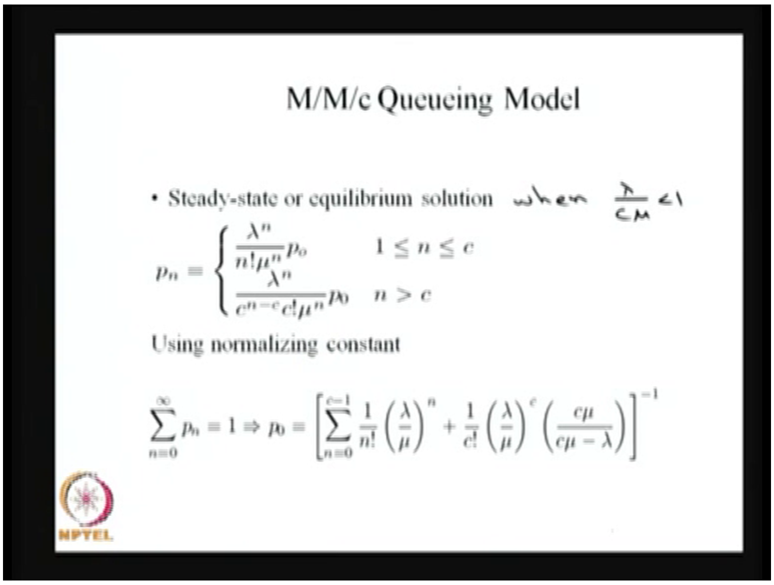

Now our interest is to find out the steady state or equilibrium solution. Since it is an infinite capacity model, if you observe the birth death process with the infinite state space, then you need a condition so that the steady state probabilities exist. So, whenever lambda by c mu is less than 1, whenever lambda by c mu is less than 1, you can find out the limiting probabilities. So, sometimes I use the letter p n sometimes I use the word pi n both are not the same.

So, you find out the steady state probability by solving by q is equal to pq is equal to 0, and the summation of a pi is equal to 1. And if you recall the birth death process the steady state the probabilities, the pi naught has the 1 divided by the series whenever the denominator series converges, then you will get the p n’s. So, either I use the p n's or pi n’s both are on the same.

So, here summation of a pi is equal to 1 and the p if you make a vector p p times a q is equal to 0 if you solve that equation. And the denominator of p naught that expression, that is going to be converges only if lambda divided by c mu is less than 1. So, therefore, whenever this condition is there the queueing system is stable also. If you put c is equal to 1 you will get the MM1 queue.

So, using the normalizing condition, you are getting the p naught and p naught is one divided by this. So, this is a series so, this series is going to be converges only if this condition is satisfied. So, by solving that equations you are getting p n in terms of p naught. And using normalizing constant you are getting a p naught.

Therefore, this is a steady state also known as the equilibrium solution for the MMc infinity model. So, here we are using the birth death process with the birth rates are lambda and the death rates are given in the this form. And use the same logic of the stationary distribution for the birth death process using that we are getting the steady state or equilibrium solution for the MMc model.