Introduction to Probability Theory and Stochastic Processes
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Lecture - 82

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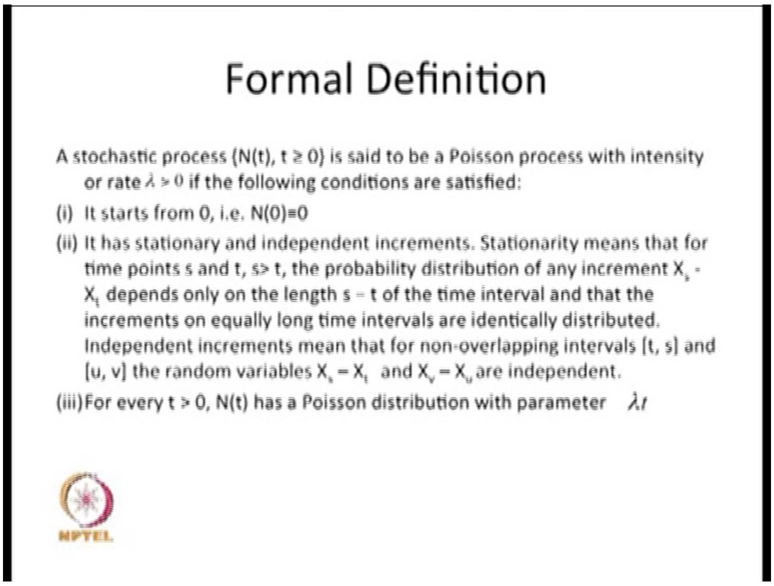

Formally, we define Poisson process as follows, a stochastic process N of t, t greater than or equal to 0 is said to be a Poisson process with the intensity or rate lambda greater than 0 if the following conditions are satisfied. First condition, it starts from 0; that is, N of 0 is equal to 0.

Second condition, the increments are stationary and independent. Stationarity means, that for time points s and t, s is greater than t, the probability distribution of any increment N of s minus N of t depends only on the length s minus t of the time interval and that the increments on equally long time intervals are identically distributed.

Independent increments means that, for any non-overlapping intervals t comma s and u comma v, the random variables N of s minus N of t and N of v minus N of u or independent for t greater than 0, N of t has a Poisson distributed random variable with the parameter lambda t.

And the difference of the random variables defined over non-overlapping intervals or independent lambda t is the cumulative rate till time t. The X i's are independent and identically distributed random variables with some distribution function g independent of the Poisson process N of t, t greater than or equal to 0. It is Markov in nature, because the 2 queues act independently and are themselves mm one queuing system which satisfies the Markov property.

Assuming that each queue behaves as the mm one queue, the details of the proof can be found in the reference books. Because q ij's are obtained by differentiating the p ij's for every t greater than 0, N of t has a Poisson distribution with the parameter lambda t. Like that, you can go for many more increments. Also, for illustration, I have made it with the 2 increments; that means, the occurrence of arrival during this non-overlapping intervals are independent and the stationary means ah, it is a time invariant only the length matters; not the actual time.

Third one, for every t N of t has a Poisson distribution with the parameter lambda t. So, the Poisson logic is coming into the fourth condition only. The first condition is start at 0, increments are stationary and the increments are independent. The third condition for fixed t, N of t is a Poisson distribution random variable with the parameter lambda t. Therefore, this stochastic process is called a Poisson process.

Now, we can relate the way we have done the derivation. We have taken care these 3 assumptions starting at time 0. 0 increments are stationary that we have taken and in increments are independent; that is, non-overlapping intervals are independent. Then, when we have derived, we are getting the distribution of the random variable N of t is a Poisson distributed random variable. Therefore, this is a Poisson process.

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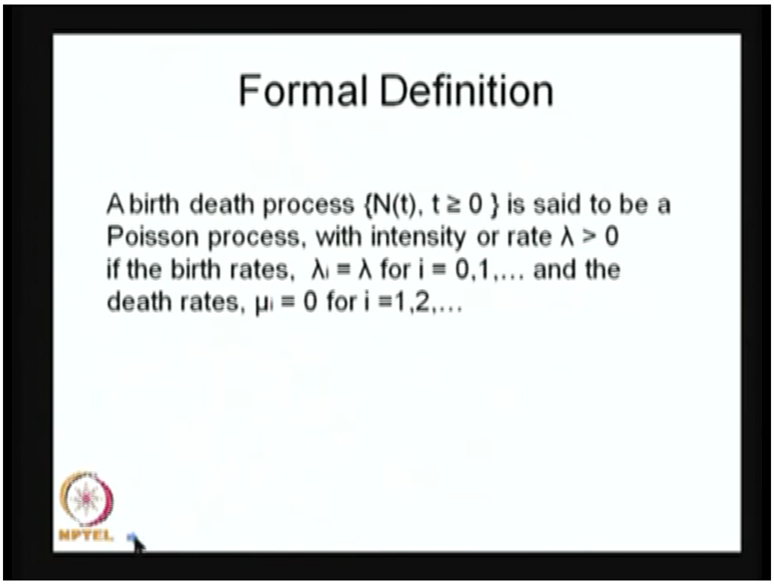

The another way of defining the Poisson process we can start with the birth death process. You know that the birth death process is a special case of a continuous time Markov chain. Also it is a special case of a sorry it is a it is a special case of Markov process also. So, you can think of a stochastic process. Then, the special case is a Markov process. Then, the special case is a continuous time Markov chain. Then, you have a special case that is a birth death process. So, you can define the Poisson process from the birth death process also.

A birth death process N of t is said to be a Poisson process with the intensity or rate lambda, if birth rates are constant for all i and the death rates are 0. You start from the birth death process with all the birth rates are same; that means, it is a special case of pure birth process in which birth rates are constant for all the states and the death rates are 0. Then also, you will get the Poisson process.

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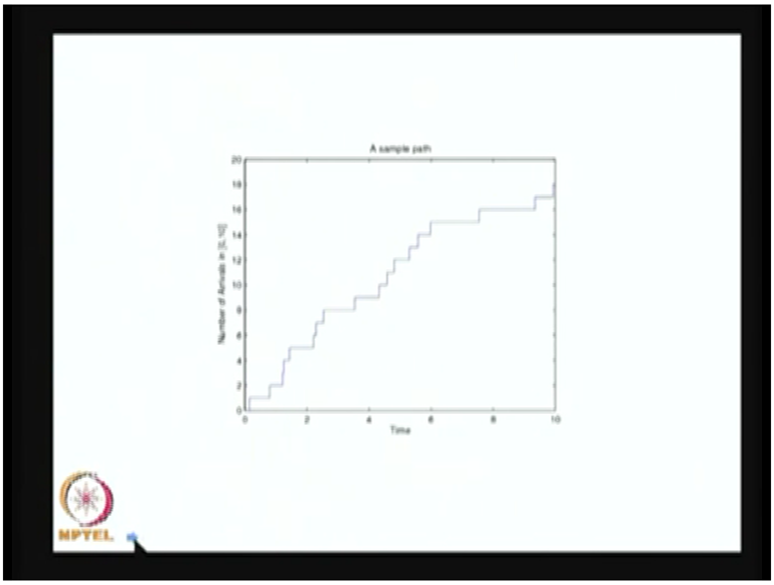

Here, I am giving us a sample path for the Poisson process. So, this is a created using the MATLAB, right the simple code of Poisson process. Then, you develop the sample path; that means, a time 0 the system at 0, at some time 1 arrival takes place therefore, the system land up 1. Therefore, the y axis is nothing, but the N of t.

So, at this time, one arrival takes place. Therefore, the number of customers in the system number of arrivals till this time that is 1. So, it is a right continuous function. The value at that point and the right limit is same as both are same which is different from the left limit of the arrival epoch arrival time of epoch. So, the system was in the state 1 till the next arrival takes place. So, suppose the arrival takes place here, then the N of t value is 2 at this time point in which arrive epoch and the right limit and so on.

So, this is the way. Therefore, the system at any time, it will be the same value or it will be incremented by only 1 unit. The Poisson process sample path will be with the 1unit step increment at any time. There is no way the 2 steps the system can move forward at a even in a very small interval of time, the system will move into the only 1 step, that you can visualize here.

Therefore, you can go back to the assumptions which we have started in the derivation N of 0 is equal to 0 in a very small interval of time at most 1 event can takes place and the difference of the random variables defined over non-overlapping intervals or independent and increments are also stationary. So, those things you cannot able to visualize in the sample path. So, this is the just one sample path over that time and the N of t.

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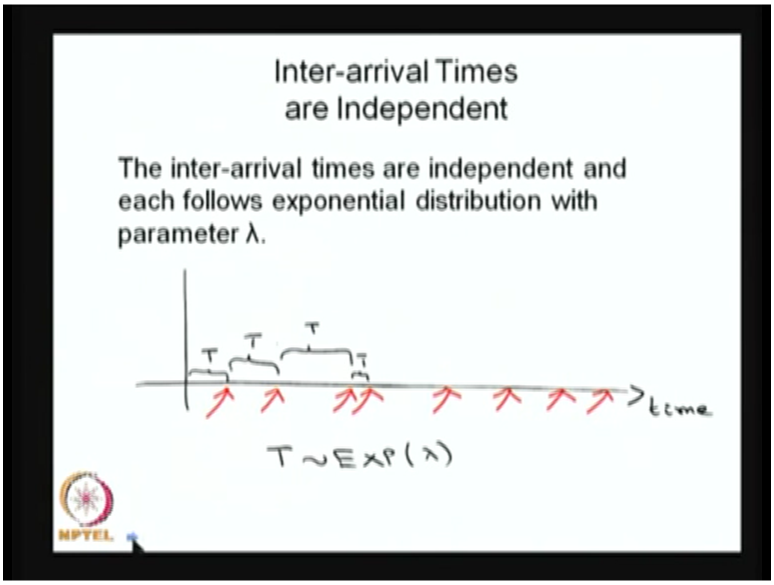

The second one, inter arrival times or independent as well as we can conclude the inter arrival times are exponentially distributed also. The inter arrival times are independent and each follow exponential distribution with the parameter lambda. What is the meaning of inter arrival times? At time 0, the system is in the state 0. First arrival occurs at this time point. Second arrival occurs this time point and the fourth, third, fourth and so on.

The inter arrival time means what is the time taken for the first arrival, then what is the interval of time taken for the first arrival to the second arrival and second to the third and so on. So, that is the inter arrival time

So, whenever you have a Poisson process; that means, the arrival of event occur over that time that follows a Poisson process, then this inter arrival time, suppose, I make it as a random variable T and those random variables going to follow exponential distribution with the same parameter lambda and all the inter arrival times also independent. That means these are all identically distributed random variable.

I can go for a different random variable label also x 1 x 2 x 3 x 4 and so on. So, all those random variables are iid random variables and each follows exponential distribution with the parameter lambda.

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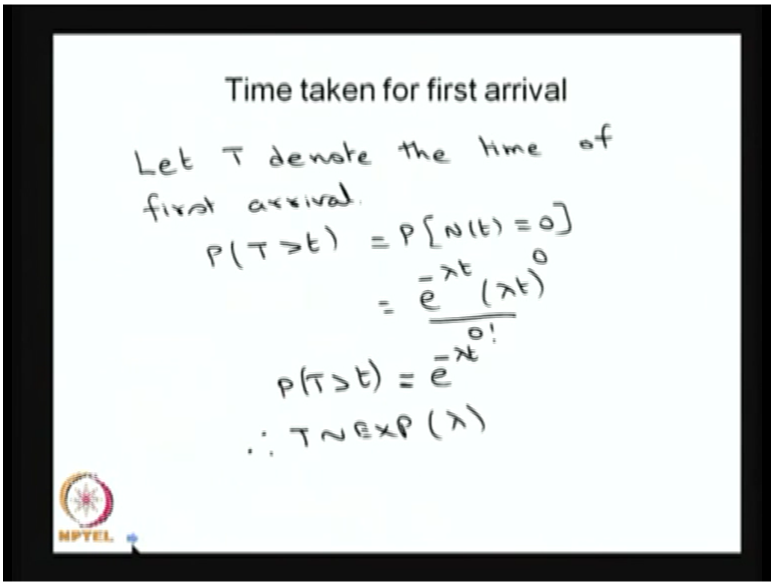

So, this can be proved easily. Let me start giving the proof for the first arrival time; that means the first 1 from 0 to the first arrival. Like that, you can go for the other arrivals also using the other properties or you can use the multidimensional random variable distribution concept and use the function of random variable and you can get the distribution also. But here, I am finding the distribution for the first arrival. So, let T denote the time of first arrival.

My interest is to find out what is a distribution of T. I know that this is going to be a continuous random variable because, it is a time. So, anytime the first arrival can occur. So, to find, since it is a continuous random variable, I can find out the CDF of the random variable or compliment CDF. So, here I am finding first the compliment CDF. Using that, I am going to find out the distribution.

Let me start with the probability that, the first arrival is going to takes place after time T. What is the meaning of that? The first arrival is going to occur after time small t; that means, till time t, there is no arrival. So, both the events are equivalent events. The probability of T greater than small t; that is same as the probability of N of t is equal to 0. That means no event takes place till time t. Because, the N of t denotes the number of arrival of customers during the interval 0 to small t both are closed 0 to 1, 0 to t. Therefore, N of t equal to 0; that means, a till time t, nobody turned up that is equivalent off the first arrival is going to takes place after t.

I do not know what is it, I do not know the distribution of T. But, I know what is the probability of N of t equal to 0. Therefore, I am writing this relation. So, once I substituted the probability mass at a 0 for the random variable N of t. Just now, we have proved that N of t for fixed t is a Poisson distribution random variable with the parameter lambda times t. Therefore, I know what is a probability mass at 0. So, substitute the probability mass function with the 0. I will get e power minus lambda t that is a compliment CDF of the random variable capital T.

Once I know the compliment CDF, I can find out the CDF from the CDF, I can compare the CDF of some standard continuous random variable. I can conclude this is nothing, but a exponential distribution with the parameter lambda because, this is a compliment CDF at time t. Therefore, it is a lambda times t.

So, I conclude the distribution of a T is exponential distribution with the parameter lambda; that means the first time of arrival this random variable that is a continuous random variable and the continuous random variable follows exponential distribution with the parameter lambda. Since, I know the increments are independent increments are stationary and so on, I can use the similar logic for inter arrival time of the this time also, then that is also going to follow exponential distribution.

Since the increments are independent. So, this is the first time and this is a second time. Therefore, the inter arrival times also going to be independent; that means, whenever you have a Poisson process; that means, the arrival occurs over that time in a very small interval maximum 1 arrival takes place. And the probability of 1 arrival in that small interval is lambda times delta t. From that, you will get the lambda. So, you can conclude that is a Poisson distribution Poisson process. So, once the arrival follows a Poisson process, the inter arrival times are exponential and independent.

So, from the Poisson process, one can get the inter arrival times are exponential distribution and independent the converse also true; that means, if some arrival follows with the inter arrival times exponential and exponential distribution and all the inter arrival times are independent. Then, you can conclude the arrival process is going to form a Poisson process; that means, arrival process or Poisson process implies the inter arrival times are exponentially distributed and are independent. Similarly, inter arrival times are independent as well as the exponentially distributed with the parameter lambda, then the arrival process is a Poisson process with the parameter lambda.