Introduction to Probability Theory and Stochastic Processes

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Lecture – 81

In this lecture we are going to discuss Poisson process and it is application. So, let me start with the Poisson process the definition, then I give some properties in the Poisson process and I also present some examples. Poisson process is very important stochastic process, whenever something happens in a some random way occurrence of some event and if it satisfies a few properties, then we can model using a Poisson process.

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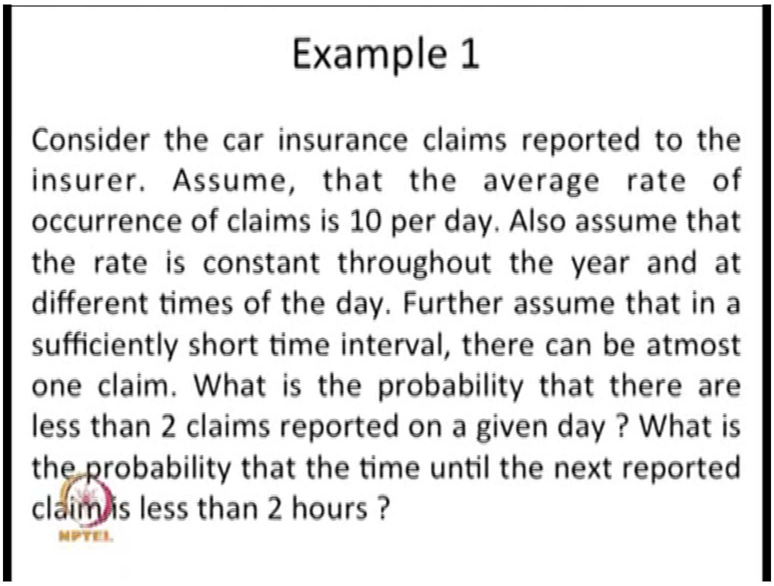

And Poisson process has some important properties whereas, the other stochastic processes have would not be satisfied with those properties. Therefore, the Poisson process is a very important stochastic process for the many modellings in a applications like, telecommunication or wireless networks or any computer systems or anything any dynamical system in which; the arrival comes in some pattern, and satisfies the few properties.

So, before moving into the actual definition of Poisson process, I am going to give one simple example; and through this example I am going to relate the Poisson processes and definition, then later I am going to solve the same example also say, example number 2 example 1, I have something else. Consider a car insurance claims a reported to insurer it need not be car insurance. You can think of any motorcar, motor insurance or any particular type of vehicle or whatever it is, assume that the average rate of occurrence of claims 10 per day. It is a average rate per day therefore, it is a rate per day the average rate is 10, also assume that this rate is a constant throughout the year and at a different times of a day.

So, even though this quantity is a average quantity there is a possibility someday there is no claim reported at all or there are some day more than some 30, 40 claims reported. And all the possibilities are there, but we make the assumption the average rate is a constant, throughout the year at the different times of a day also. Further assume that in a sufficiently short time interval there can be at most one claim.

Suppose, you think of a very small interval of like 1 minute or 5 minutes or whatever very small quantity comparing to them because here I have given the average rate is a 10 per day therefore, whatever the time you think of a very negligible in that, the probability of or it is sufficiently small interval of time; there is a possibility of only maximum one claim can be reported. The question is, what is the probability that there are less than 2 claims are reported on a given day. What is a probability that less than 2 claims reported; means, that what is a probability that in a given day, either no claim or one claim.

Also we are asking the second question: what is the probability that; the time until the next reported claim is less than 2 hours. Suppose, sometime one claim is reported what is the probability that the next time is going to be reported before 2 hours. We started with this problem the car insurance claims reported therefore, the claims is nothing but some event and these events are occurring over the time. Suppose, you make the assumption of sufficiently small interval of time at most one claim can happen and the average rate of occurrence of claim is a constant throughout the time.

So, with this assumption one we one we can think of a sort of a arrival process, a pure birth process satisfying some condition and that may lead into Poisson process. So, this same example we are going to consider it again also. Now, I am going for definition of a Poisson process.

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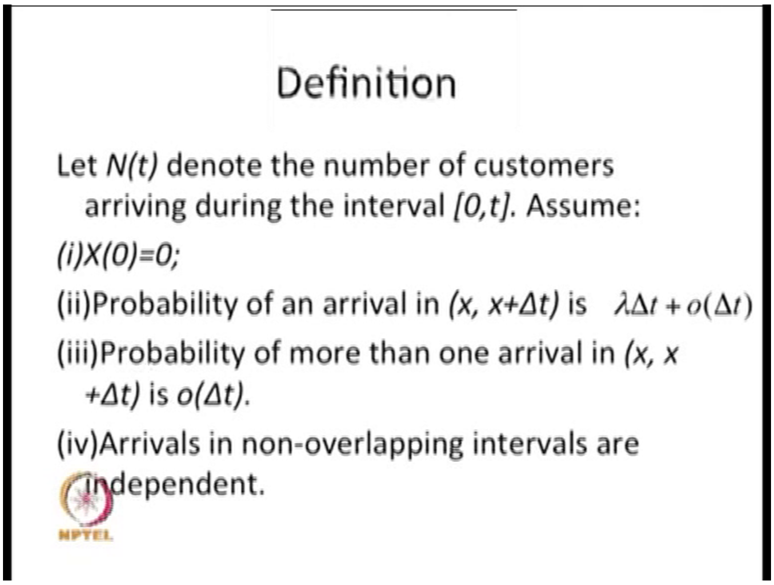

How one can derive the Poisson process. Poisson process is a stochastic process with some conditions. So, how one can derived the Poisson process, for that let me start with the random variable N of t that denotes the number of customers arriving during the interval 0 to time t; that means, how many arrivals are takes place in the interval 0 to t; that means, a for fixed t N of t is a random variable over the time this N of t collection that is, a stochastic process. I am making a some 4 assumptions with these assumptions I am going to conclude the N of t is going to be a stochastic process. The first assumption not x of 0 N of 0 is equal to 0 at time 0 the number of customer's is 0 N of 0 is equal to 0 it is a wrong N of 0.

Second one the probability of a arrival in interval x to x plus delta t that is, a lambda times delta t where lambda is strictly greater than 0; that means, probability that I have only one arrival is going to takes place in a interval of delta t. That probability is a lambda times delta t for a very, very small interval delta t, it is independent of x; that means, it is a increments are stationary that, property I am going to introduce in this assumption.

The probability of more than one arrival in the interval x 2 x plus delta t is negligible; that means, at most a maximum one arrival can occur in a very small interval of time that is the assumption, that I am specifying in third one. The 4th assumption arrivals in non overlapping intervals are independent; that means, if the arrival occurs in a some interval and another some non overlapping interval then those arrivals are going to be from independent; that means, there is no dependency over the non overlapping intervals arrivals going to occur or not.

So, with these 4 assumptions N of 0 is equal to 0 and probability of one arrival is lambda times delta t in a small interval more than one arrival occurrence in interval delta t, where delta t is very small that is that probability is negligible and non-overlapping intervals arrival or independent.

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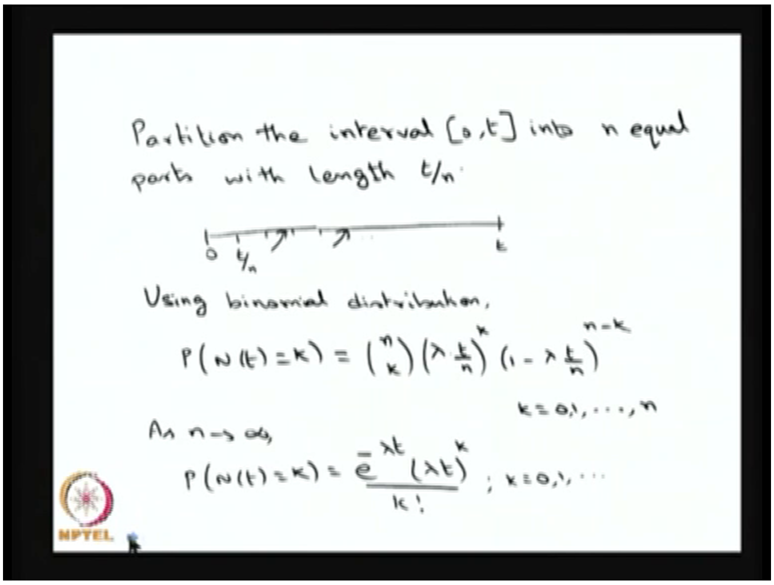

So, with this derivation I am going to find out the distribution of N of t. To find the distribution of N of t, first I am doing I am partitioning the interval 0 to t into N equal parts with the length t divided by N. The way I use the way I partition the interval 0 to t into n pieces such that, a t by n is going to be a very small interval so; that means, I have to partition that interval 0 to t in such a way that; the t by n is going to be as small as therefore, I can use those assumption of a probability of occurring one arrival in that interval of length t by n that, probability is a lambda times t by n.

And probability of not occurring event in that interval t by n is; 1 minus lambda time t by n. So, I can use those concepts for that I have to partition the interval 0 to t into n parts with a sufficient larger N therefore, t by n is going to be smaller. Now since, I partition this interval into n pieces n parts; I can think of at each parts I can think of a Binomial or Bernoulli distribution.

At each pieces therefore, all the non overlapping intervals occurrence are independent therefore, I can think of it is accumulation of n independent Bernoulli trials. Since it is a n independent Bernoulli trials for each intervals t by n of length t by n therefore, the total number of event occur in the interval 0 to t, by partitioning into N equal parts.

This is sort of what is the probability that k event occurs in the interval 0 to in the time duration 0 to t as a n partition. So, out of n equal parts what is the probability that a k events occur in the interval 0 to t that is nothing but since it is a each interval is going to form a Bernoulli distribution with the probability P is lambda times t by n therefore, the total number is going to be binomial distribution with the parameters n and p, where P is lambda times t by n.

Therefore, this is the probability mass function of a k event occurs out of N equal parts therefore, n c k lambda times t by n power k 1 minus lambda times t by n power n minus k. Now, there running index power k goes from 0 to n; that means, there is a possibility no event takes place in the interval 0 to t or maximum of n interval n even takes place in all n intervals. So, this is for a sufficiently large n such that; the t by n is smaller. We take n tends to infinity to understand the limiting behavior of this scenario as the partition becomes finer.

Now, I can go for n tends to infinity, what will happen? As n tends to infinity, if you do the simplification here as n tends to infinity; that simplification I am not doing in this presentation has a limit n tends to infinity, the whole thing will land up e power minus lambda t lambda t power k by k factorial. Now the k running index is 0, 1, 2 and so on. This you can use the concept the binomial distribution has n tends to infinity and the P tends to 0 you are n into P becomes a lambda so, that will give the Poisson distribution.

The limiting case of a binomial distribution is a Poisson distribution. So, using that logic this binomial distribution mass has n tends to infinity this becomes a Poisson distribution mass function. So, this is nothing but the right hand side is the probability mass function for a Poisson distribution with the parameter lambda times t. And this is a random variable N of t for fixed t therefore, for fixed t N of t is a Poisson distributed random variable with the parameter lambda times t where lambda is greater than 0.

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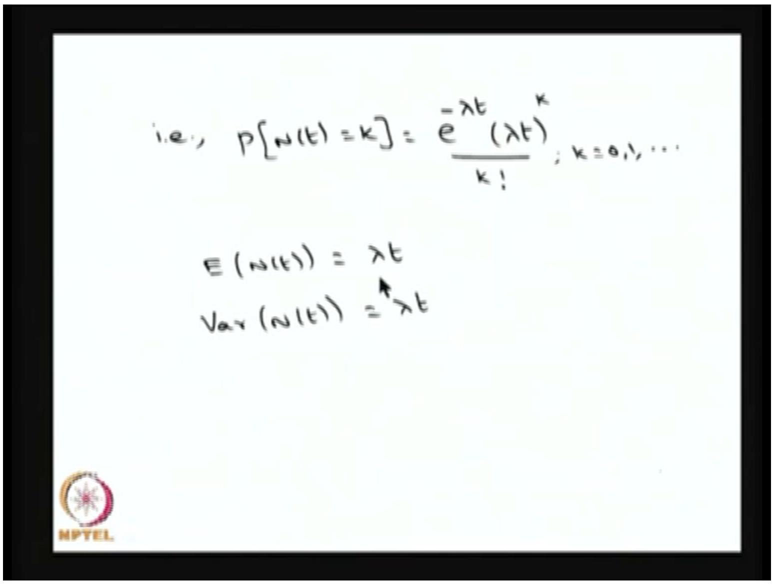

Therefore, we can conclude; the stochastic process related to the N of t for fixed t N of t is Poisson distribution. Therefore, the stochastic process N of t over the t greater than or equal to 0 that is, nothing but a Poisson process. So, from the Poisson distribution, we are getting Poisson process because each random variable is a Poisson distributed with the parameter lambda times t therefore, that collection of random variable is a Poisson process with the parameter lambda t. Since it is a Poisson distributed random variable for fixed t, you can get the mean and variance and all other moments also; by using the probability mass function of N of t.