**Introduction to Probability Theory and Stochastic Processes**
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**Lecture – 80**

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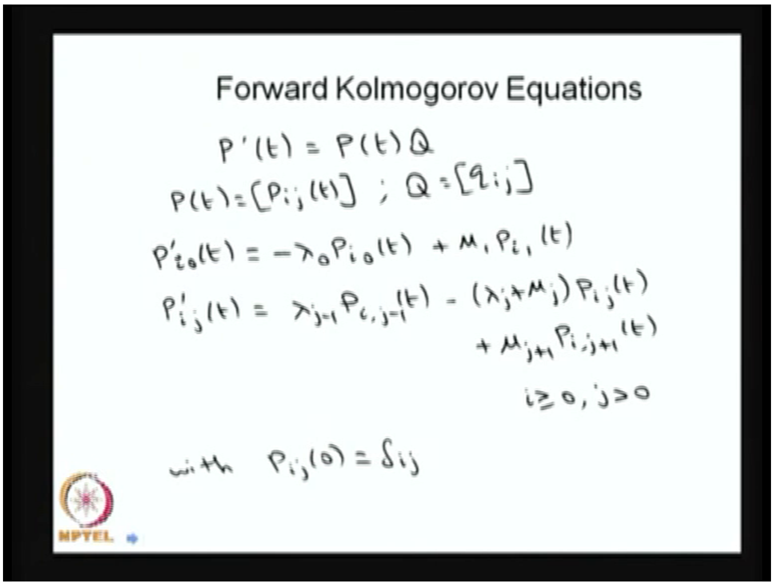


We are discussing the forward Kolmogorov equation for a special case of Continuous time Markov Chain that is a birth death process. For a birth death process, the Q matrix is a tri diagonal matrix. Therefore, you will have a the equations from the forward Kolmogorov equation, you will have a only 2 terms in the right hand side for the first equation And you will have only 3 terms; the diagonal element and 2 off diagonal elements.

Therefore, the second equation, one can the first equation, one can discuss first the P dash of i comma 0; that is nothing, but the system is not moved from the state 0. Moving from the state 0 that rate is lambda naught.

Therefore, not moving minus lambda naught times the probability and a or the system can come from the state 1 with the rate mu 1. Therefore, mu 1 times P i comma 1 of t. For all other equations, either system comes from the previous state with the rate lambda j minus 1 or it comes from the forward one state with the rate mu j plus 1 or not moving anywhere.

So, these are all the all possibilities therefore, with these 3 possibilities. you have a 3 terms in the right hand side and that is the net rate for any state j. So, if you solve this equation with this initial condition Kolmogorov delta i comma j, you will have the solution of a P i comma j.

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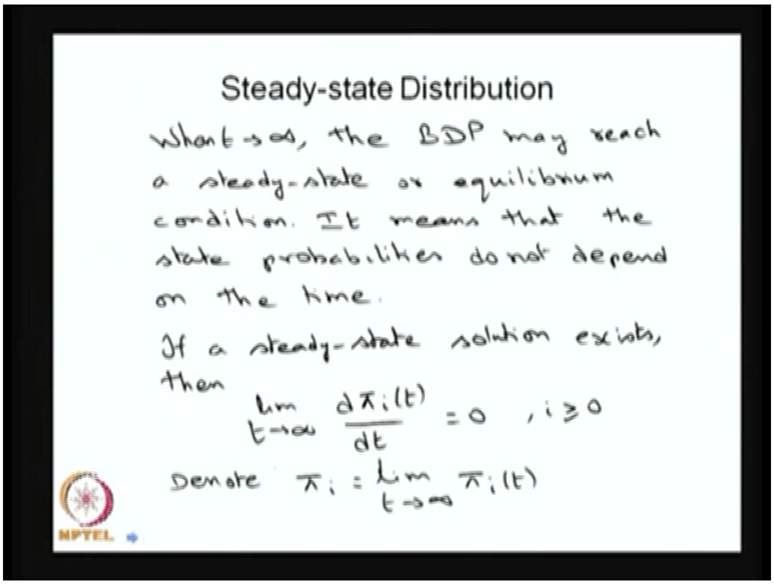


Here I am discussing the steady state distribution. The way I have discussed the limiting distribution, that is, a limit t tends to infinity, probability of i comma j of t insist and then, it is called the limiting distribution and the stationary distribution is nothing, but for the DTMC, it is a pi P is equal to P. Summation of pi is equal to 1 for the CTMC it is pi q is equal to 0 and the summation of pi i is equal to 1 that is going to be steady state distribution stationary distribution.

Now, I am discussing the steady state distribution; that is nothing but, when t tends to infinity, the birth death process may reach steady state or equilibrium condition; that means, the state probabilities does not depend on time. That is the meaning of a steady state distribution; as a t tends to infinity, whenever we say the birth death process reaches a steady state or equilibrium, that state probability does not depend on time; that means, if a steady state solution exist since the time depend.

Since the state probability is does not depend on time t. The derivative of the time dependent state probability at time t that derivative at t tends to infinity it becomes 0. If the steady state solution exist, since the state probabilities does not depend on time t as a t tends to infinity, I can write as a pi is a limit a tends to infinity of pi i of t.

So, this is different from the way we discuss the earlier that conditional probability P i j of t. But using P i j of t, one can find out what is pi i j of pi i of t; that is nothing, but the pi i of t; that I have given in the first lecture for the CTMC.

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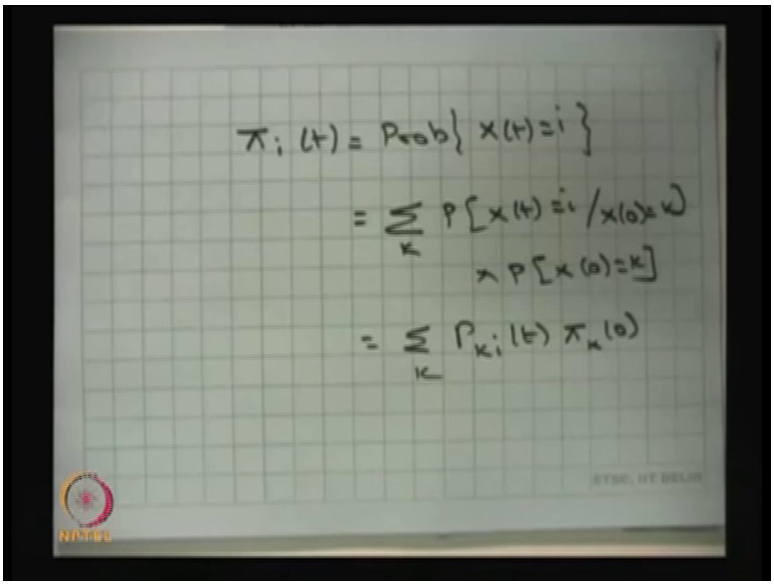


The pi i of t that is nothing, but what is the probability that the system will be in the state i at times. That is same as what is the probability that the system will be in the state i given that, it was in the state some k at time 0 multiplied by what is the probability that it was in the state k at times k.

That is nothing, but summation of k and this is nothing, but the transition probability and this is nothing, but the initial probability vector, very good. So, using P k i of t or P i j of t, that is a conditional probability, one can get the unconditional probability. This is nothing, but the distribution of X of t. So, this is a probability mass function; probability mass at state i.

So, now, what I am defining? Whenever the steady state distribution exist; that means, it is independent of time t. Therefore, as a t tends to infinity, the pi of t can be written as the pi i and whenever the steady state solution exists, I can use a limit t tends to infinity; the derivative of a pi i of t that is going to be 0.

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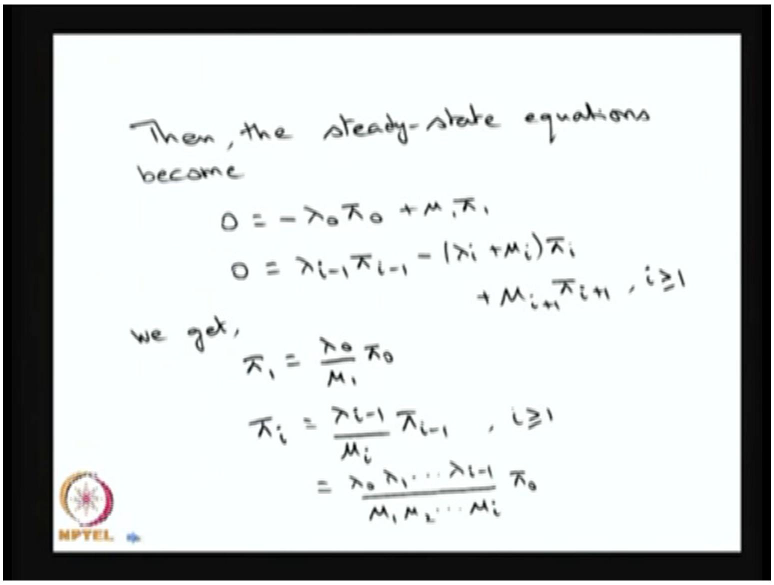


Therefore, I am going to use these 2 to get the steady state probabilities for the birth death process. Since, a as a t tends to infinity, the derivative of P i j of t is equal to 0. Therefore, all the left hand side the forward Kolmogorov equation that is going to be 0. The right hand side, you will have a as a t tends to infinity the pi i of t, that can be written as the pi 0 and pi 1.

So, the way we write the conditional probability for P i j with the Kolmogorov forward equation, you can write the similar equation for the unconditional probability pi i's also. So, now, I am putting the left hand side 0s. Because of the this condition limit t tends to infinity, the derivative is equal to 0 and the right hand side, I am using as a t tends to infinity, this probability is nothing, but the pi i's.

Therefore, it is going to be minus lambda naught times pi naught plus mu 1 times pi naught. And all other equation has a 3 terms and this homogeneous equation and you need a one normalizing condition.

So, from this homogeneous equation, I can get recursively pi i's in terms of pi naught. So, from the first equation, I can get a pi 1 in terms of pi naught and the second equation, I can get a pi 2 in terms of first pi 1. Then, I can get a pi 1 in terms of pi naught. Therefore, recursively I can get a pi i's in terms of pi naught for all i greater than or equal to 1.

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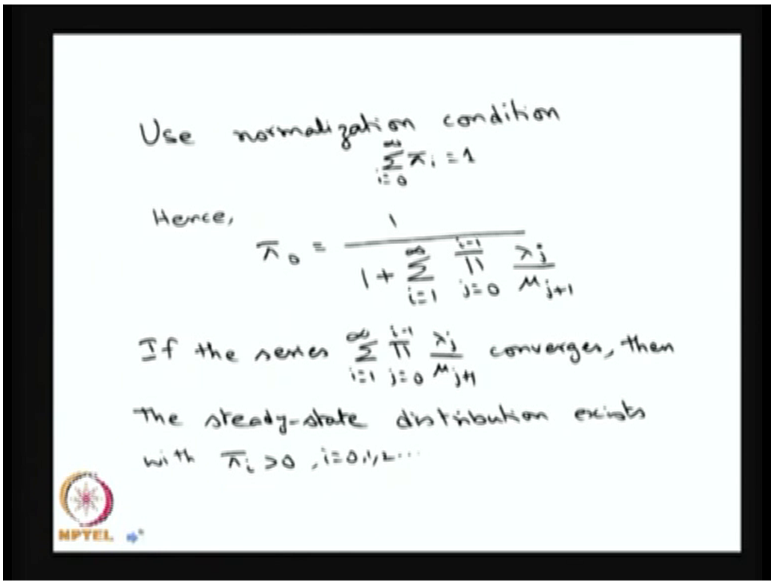


Now, I can use a normalizing condition. Summation of pi i is equal to 1. Therefore, I will get a pi naught is equal to 1 divided by summation of these many terms.

In the product form, since we need a steady state probabilities and all the pi i's are in terms of pi naught, as long as a the denominator is converges, you will have a pi 0 is greater than 0. Once a pi 0 is greater than 0, then we will get all the pi i's if the summation of pi i is equal to 1.

So, whenever these series converges, then I will have a steady state distribution with the positive probability and the summation of probability is going to be 1.

So, this is the condition for a steady state distribution for a birth death process. Because, we started with the birth death process, a forward Kolmogorov equation using these 2 conditions, we have simplified into this form and use a normalizing condition and get the pi naught.

As long as long as the summation is a or the series is converges, then we will have the steady state if the series diverges; that means, a by substituting the values for the lambda i's and mu i's. And if the series the denominator series diverges, then the pi naught is going to be 0. In turn, all the pi i's are is equal to 0. Therefore, the steady state distribution will not exist if the denominator series diverges.

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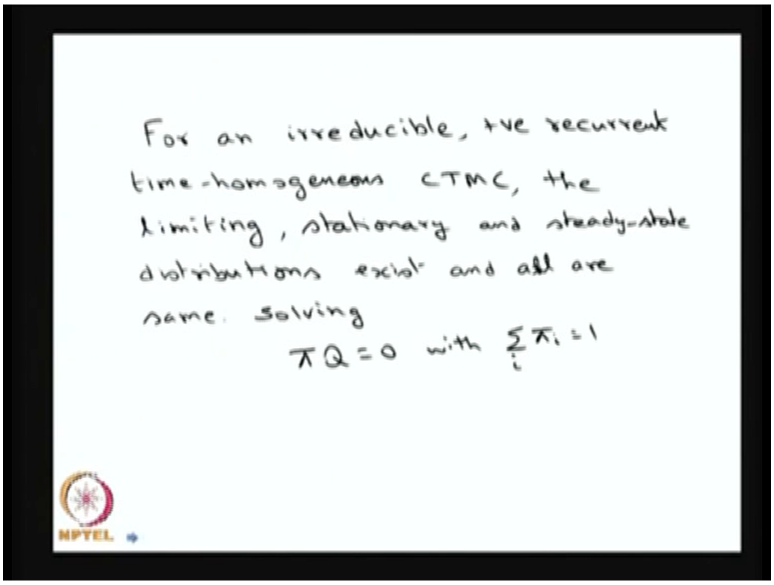


I am going to give a one simple result. For a reducible positive recurrent a time homogeneous CTMC, we know that the limiting distribution exist stationary distribution exist. Now, I am including the steady state distribution also exist. I have given for a steady state distribution for the birth death process; not for the CTMC. But here, I am giving the result for the CTMC. All the 3 distributions exist.

And all are going to be same. Whenever the CTMC is a time homogeneous irreducible positive recurrent, all these 3 distributions are seen and one can evolve it. One can solve these 2 equations pi Q is equal to 0 and with the summation of pi i's equal to 1, you can get the limiting distribution stationary distribution or steady state or equilibrium distribution.

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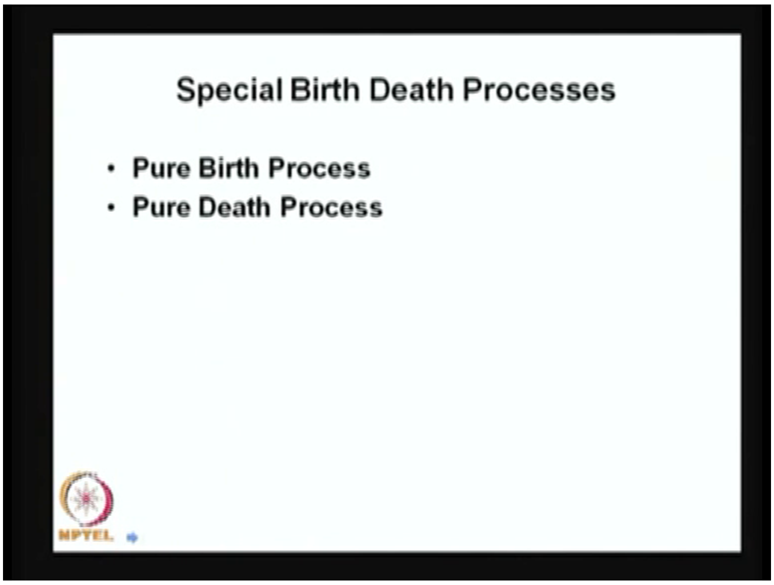


As a special case of birth death process, I am going to discuss a these 2 process in this lecture.

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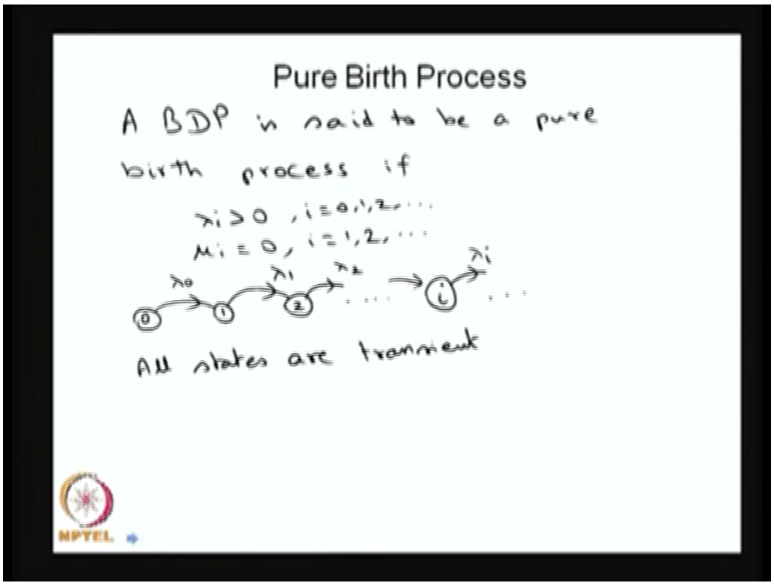


Whenever we say the birth death process is a pure birth process that means, all the death rates are going to be 0. We started with the birth death process with the only lambda is; are greater than 0 and the mu i i's are going to be 0. Then, it is going to be called it as a pure birth process.

There is a one special case of pure birth process with the lambda i's are going to be constant; that is, lambda, that is a Poisson process, that I am going to discuss in the next lecture. And in this pure birth process, this lambda is are the function of i. Here, all the states are transient states.

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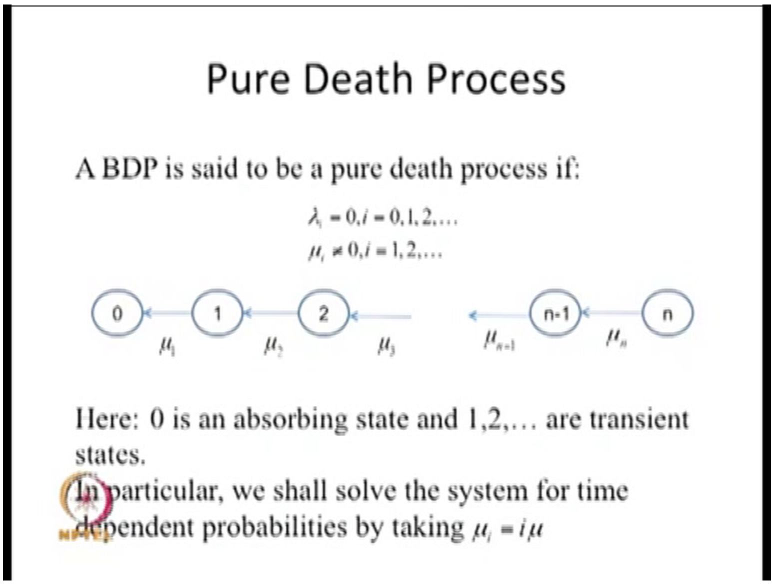


Here I am discussing the pure death process. A birth death process is said to be a pure death process if the birth rates are 0 and the death rates are non 0. In particular, we shall obtain the time dependent probabilities of a pure death process in which the death rates mu i's or equal to i times mu. As I given the example, it as a fourth example in the birth death process. This state 0 is a observing barrier.

Therefore, the state 0 is a observing state and all other states are going to be transient state and here, the limiting distribution exist and the one can also find the time dependent probabilities for this model.

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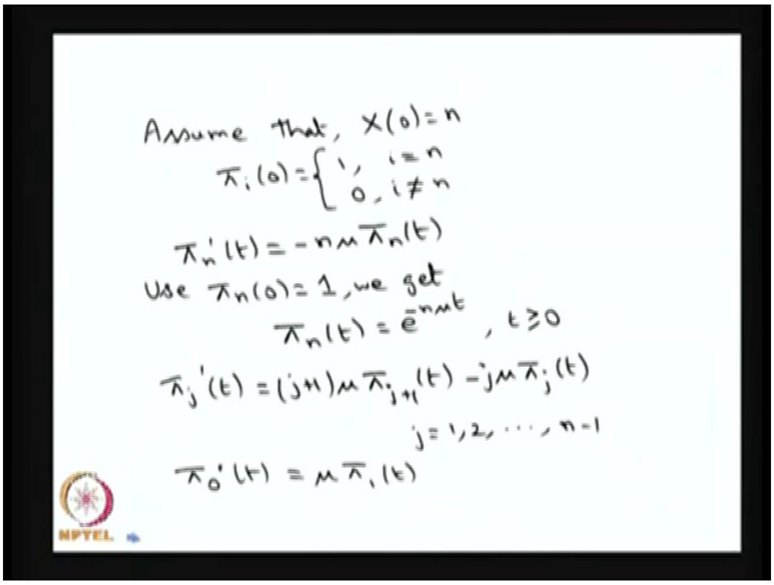


Suppose, you start with the assumption, the system at time 0 in the system is in the state n at time 0.

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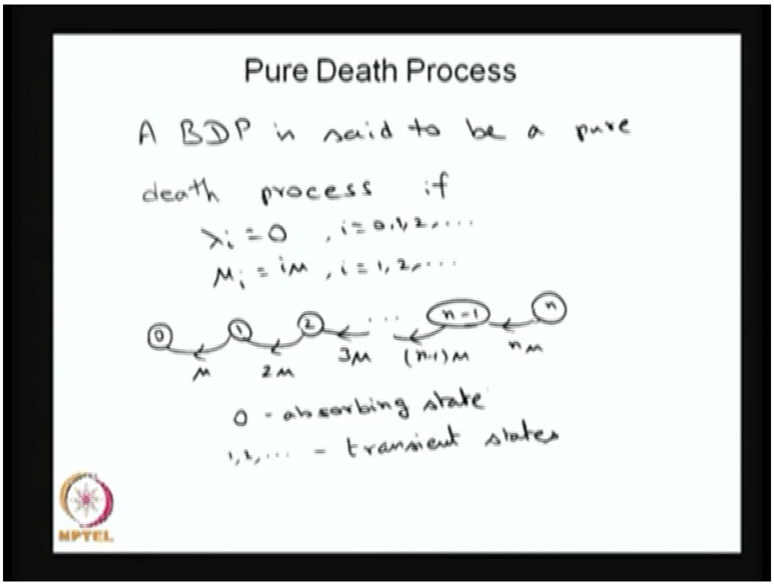


The systems in the state n at time 0 with that assumption, I can frame the equation; that is a pi and dash of t is equal to minus n times mu of pi n of t; that means, that the rate in which the system is move in the state n. That is nothing, but the not moving to the state n minus 1 with the rate n minus n times mu; therefore, the equation for the state n; that is a pi n dash of t that is equal to not moving from the state n.

Therefore, minus that outgoing rate that is n times mu being in the state is n. Therefore, pi n of t, I can use the initial condition pi n of 0 is equal to 1. So, I will get pi n of t. For the second equation, I have to go for what is the equation for the state n minus 1. So, the pi n minus 1 dash t; that is nothing, but either the system come from the state n or not, moving from the state n minus 1; therefore, the system coming from the state n.

That is a n mu times the system being the state n minus n minus 1 times mu pi n minus 1 of t. So, we will have a 2 terms in the right hand side coming from the one forward state or not moving from the same state.

So, we will have a 2 terms for j is equal to 1 to n minus 1. For the last state, that is the state 0, the systems come from the state 1. Since the state 0 is observing states, there is no second term. So, it is going to be mu times pi n of t.

So, you know pi n of t. Use the pi n of t in the equation for the n minus 1 and get the pi n minus 1. Like that, you find out a tail pi 1. Use the pi 1 to get the pi 0 of t.

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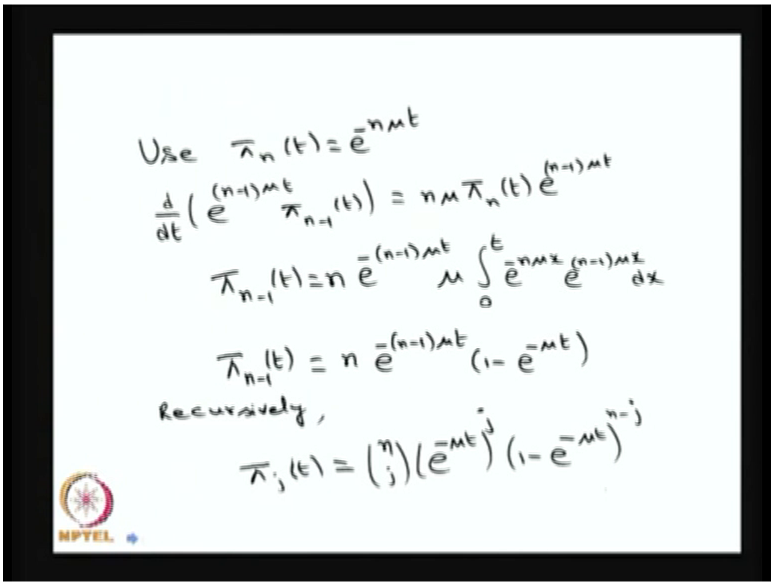


Use the recursive way. So, using the recursive way, you will get the pi j of t is equal to n C j. Combination n C j and e power minus mu times t power j. This a survival probability of system being in the state and 1 minus e power minus mu of t n minus j.

Suppose, the system being in the state j; that means, from the state n, this many combination would have come and the survival probability is e power minus mu times t and that power.

So, this is nothing, but the probability P power j and the 1 minus P power n minus j. Therefore, this pi j follows a binomial distribution with the survival probability e power minus mu t being in the state j.

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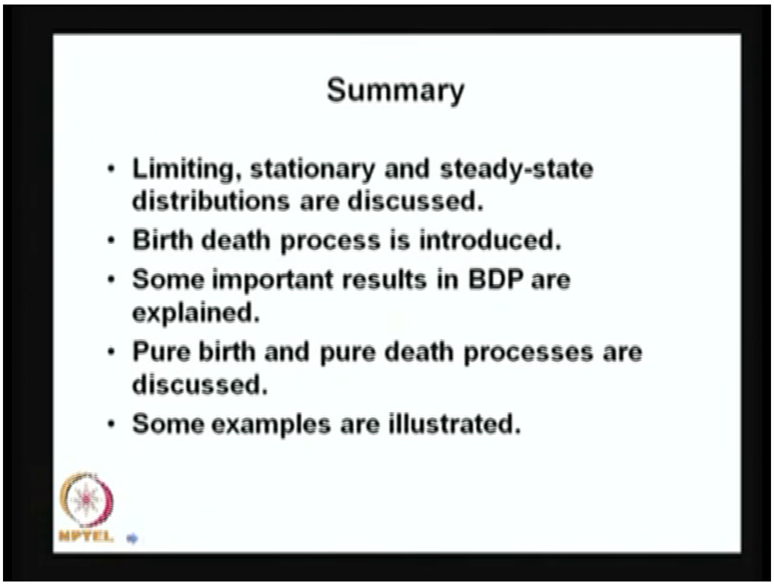


So, for the few death process, I have explained the time dependent probabilities of being in the state j that is a unconditional probability. So, with this, the summary of this lecture is, I have discuss the limiting stationary and a steady state distributions. I have introduced a birth death process.

Some important results also discussed and at the end, I have discussed the pure birth and pure death processes also. In the next lecture, I am going to explaining the important pure birth process that is a Poisson process.

Thanks.